RELATIVISTIC MAGNETOFLUIDS AND SYMMETRY MAPPINGS III

by G. PRASAD *and* B. B. SINHA, *Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi* 221005

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In this paper certain symmetry mappings belonging to the family of contracted Ricci collineations admitted by the relativistic magnetofluid space-times for the time-like symmetry vectors along the fluid flows and related conservation expressions are investigated. Some theorems of physical interest have been also explored in the area of local conservation laws involving FCRC quasi-symmetry properties admitted by the imperfect magnetofluid space-times.

1. INTRODUCTION

The study of the behaviour of relativistic magnetofluids has become the fashion of the day and is physically important in several contexts : cosmology and late stages of stellor collapse. Recently Esposito and Glass (1977) have established the conservation of magnetic energy for shear free expanding motions using the relativistic magnetohydrodynamics (RMHD) field equations given by Lichnerowicz (1967).

The purpose of this paper is to demonstrate the conservation expression obtained by Esposito and Glass (1977) and other conservation expressions which have not been found by them employing symmetry methods developed by Davis (1974), Davis *et al.* (1976) and Oliver and Davis (1976). Norris *et al.* (1977) have investigated the time-like and space-like family of contracted Ricci collineations (FCRC) symmetry mappings in the context of perfect fluid including non-interacting electromagnetic fields. This paper is devoted to the study of FCRC symmetry mappings admitted by the imperfect magnetofluid space-times for the time-like symmetry vectors to complete the series of papers (Prasad 1978 a, b) in which the space-like FCRC symmetry mappings are investigated.

2. IMPERFECT MAGNETOFLUID SPACE-TIMES AND FCRC

Norris *et al.* (1977) have defined the FCRC admitted by the four dimensional Riemannian manifold V_4 by the conditional relations

$$\mathop{\mathcal{L}}_{\xi} R_{ij} = H_{ij}, \; g^{ij} H_{ij} = 0 \qquad \ldots(2.1)$$

where H_{ij} is any trace-free symmetric tensor that is not identically equal to $\underset{\xi}{\mathcal{L}} R_{ij}$. Here $\underset{\xi}{\mathcal{L}}$ denotes the operation of Lie differentiation with respect to the vector ξ^i. When a vector ξ^i satisfying (2.1) can be determined to within a multiplicative constant for a particular H_{ij} in a given space-time then we say that given space-time admits FCRC symmetry property. When a particular choice of H_{ij} in a given space-time does not permit the determination of the vector ξ^i [e.g., when ξ^i is only determined to within an arbitrary multiplicative function f satisfying $\xi^i f_{,i} = 0$] then we say that given space-time admits an FCRC quasi-symmetry property.

It can deliberately be shown that the conservation expression (Davis *et al.* 1976)

$$(\sqrt{-g}R^i_j \xi^i)_{;i} = 0 \qquad \text{...(2.2)}$$

holds for both FCRC symmetry and quasi-symmetry properties.

The stress-energy-momentum tensor for a thermally conducting, compressible fluid with infinite electrical conductivity and constant magnetic permeability is given by (Prasad 1978a)

$$T_{ij} = (\overset{*}{\rho} + p^*)u_iu_j - p^*g_{ij} + q_iu_j + q_ju_i - \mu h_ih_j) \qquad \text{...(2.3)}$$

where

$$\overset{*}{\rho} = \rho + \tfrac{1}{2}\mu \mid h \mid^2, \; p^* = p + \tfrac{1}{2}\mu \mid h \mid^2. \qquad \text{...(2.4)}$$

Here ρ^*, p^*, q_i, h_i and μ denote the total enery density of the imperfect magnetofluid, the total pressure, the heat-flux vector, the magnetic field vector and the magnetic permeability respectively. The velocity of light is assumed to be unity. The heat flux vector is given by the expression (Eckart 1940)

$$q^i = K(T_{,j} - TDu_j)\,\gamma^{ij} \qquad \text{...(2.5)}$$

where K is the coefficient of heat conduction, T is the rest temperature and γ^{ij} is the projection tensor.

The field equations (Lichnerowicz 1967) governing the behaviour of relativistic imperfect magnetofluids are

$$R_{ij} - \tfrac{1}{2}Rg_{ij} = kT_{ij} \qquad \text{...(2.6)}$$

where k is a gravitational constant and R_{ij} is the usual Ricci tensor.

By virtue of (2.3) and (2.6), we have

$$R_{ij} = k\,\{\mu_0u_iu_j - \mu_1\gamma_{ij} + q_iu_j + q_ju_i - \mu h_ih_j\} \qquad \text{...(2.7)}$$

where

$$2\mu_0 = \rho + 3p + \mu \mid h \mid^2, \;\; 2\mu_1 = \rho - p + \mu \mid h \mid^2.$$

If we consider time-like symmetry mapping vectors $\xi^i = \varphi u^i$, i.e. symmetry mappings along the direction of the magnetofluid flows then (2.7) assumes the form:

$$
\begin{aligned}
k^{-1} \underset{\xi}{\mathcal{L}} R_{ij} = {} & A(u_i u_j - \tfrac{1}{4} g_{ij}) + 2\mu_0 u_{(i} \gamma^{k}_{j)} (\varphi D u_k + \varphi_{,k}) \\
& - 2\mu_1 \varphi \sigma_{ij} + 2 u_{(i} \gamma^{k}_{j)} \varphi D q_k - 2\mu\varphi h_{(i} \overset{*}{\gamma}{}^{k}_{j)} D h_k \\
& + 2 q_{(i} \hat{\gamma}^{k}_{j)} (\varphi D u_k + \varphi_{,k}) + 2\varphi \{ q^k \sigma_{k(i} u_{j)} \\
& + q^k \omega_{k(i} u_{j)} \} - 2\mu\varphi h^k \omega_{k(i} h_{j)} \\
& + (\varphi\theta + \varphi_{,k} u^k)(q_i u_j + q_j u_i) \\
& - 2(\varphi_{,k} q^k + \varphi q^k D u_k)(a_i a_j + \tfrac{1}{4} g_{ij}) \\
& + \tfrac{1}{2} (\mu_0 \varphi u^k)_{;k} g_{ij} + \tfrac{1}{2} (\varphi q^k)_{;k} g_{ij} - 2\mu\varphi (D \ln | h | V^{1/3}) h_i h_j \\
& - 2\mu\varphi h^k \sigma_{k(i} h_{j)} \qquad \ldots(2.8)
\end{aligned}
$$

where

$$A = (2\mu_0 \varphi u^k)_{;k} - \tfrac{4}{3} (2\mu_0)^{1/2} \varphi \{ (2\mu_0)^{1/2} u^k \}_{;k}$$

$$\overset{*}{\gamma}{}^{k}_{j} = \delta^{k}_{j} - u^k u_j + n^k n_j, \quad \hat{\gamma}^{k}_{j} = \delta^{k}_{j} - u^k u_j + a^k a_j.$$

Here n^i is unit magnetic field vector and a^i is unit heat flux vector.

We observe that the last four terms on the right hand side of (2.8) are not trace-free. Let us assume

$$D \ln | h | V^{1/3} = 0, \text{ as } \mu \neq 0, \varphi \neq 0 \qquad \ldots(2.9a)$$

where D is absolute derivative along the stream lines, and

$$h^k \sigma_{ki} = 0 \text{ as } \mu \neq 0, \varphi \neq 0 \qquad \ldots(2.9b)$$

which represents the differential rotation with the magnetic field. The magnetic field vector is an eigen vector of the shear tensor with zero eigen value. The relation (2.9*a*) is of astrophysical significance since it reveals that the magnetic intensity will increase tremendously for the contracting proper volume of the magnetofluid element. The remaining two terms out of the four last terms on the right hand side of (2.8) vanish due to (2.2) and represents conservation expressions in the domain of imperfect magnetofluids as follows :

$$\{\varphi(\mu_0 u^k + q^k)\}_{;k} = 0 \qquad \ldots(2.9c)$$

with $\mu_0 = \tfrac{1}{2} (\rho + 3p + \mu | h |^2).$

To discuss FCRC admitted by the imperfect magnetofluid space-times we focus our attention on the skeleton structures formed by the FCRC quasi-symmetry properties because the role of the skeleton structures is important for the study of actual symmetry properties. Following Norris *et al.* (1977) we write only those types of different choices for the tensor H_{ij} which may be of physical interest particularly as below :

$$
\begin{aligned}
k^{-1} \overset{1}{H}_{ij} &= \Lambda(u_i u_j - \tfrac{1}{4} g_{ij}) + 2\mu_0 u_{(i}\gamma_{j)}^{k} (\varphi D u_k + \varphi_{,k}) \\
&- 2\mu_1 \varphi \sigma_{ij} + 2u_{(i}\gamma_{j)}^{k} \varphi D q_k - 2\mu\varphi h_{(i}\overset{*}{\gamma}{}_{j)}^{k} D h_k \\
&+ 2q_{(i}\hat{\gamma}_{j)}^{k} (\varphi D u_k + \varphi_{,k}) + 2\varphi \{q^k \sigma_{k(i} u_{j)} + q^k \omega_{k(i} u_{j)}\} \\
&- 2(\varphi_{,k} q^k + \varphi q^k D u_k)(a_i a_j + \tfrac{1}{4} g_{ij}) \\
&+ (\varphi\theta + \varphi_{,k} u^k)(q_i u_j + q_j u_i) \qquad \ldots(2.10)
\end{aligned}
$$

$$
\begin{aligned}
k^{-1} \overset{2}{H}_{ij} &= \Lambda(u_i u_j - \tfrac{1}{4} g_{ij}) - 2\mu_1 \varphi \sigma_{ij} + 2u_{(i}\gamma_{j)}^{k} \varphi D q_k \\
&- 2\mu\varphi h_{(i}\overset{*}{\gamma}{}_{j)}^{k} D h_k + 2\varphi \{q^k \sigma_{k(i} u_{j)} + q^k \omega_{k(i} u_{j)}\} \\
&- 2\mu\varphi h^k \omega_{k(i} h_{j)} + (\varphi\theta + \varphi_{,k} u^k)(q_i u_j + q_j u_i) \\
&- 2(\varphi_{,k} q^k + \varphi q^k D u_k)(a_i a_j + \tfrac{1}{4} g_{ij}) \qquad \ldots(2.11)
\end{aligned}
$$

$$
\begin{aligned}
k^{-1} \overset{3}{H}_{ij} &= \Lambda(u_i u_j - \tfrac{1}{4} g_{ij}) + 2\mu_0 u_{(i}\gamma_{j)}^{k} (\varphi D u_k + \varphi_{,k}) \\
&+ 2u_{(i}\gamma_{j)}^{k} \varphi D q_k - 2\mu\varphi h_{(i}\overset{*}{\gamma}{}_{j)}^{k} D h_k \\
&+ 2q_{(i}\hat{\gamma}_{j)}^{k} (\varphi D u_k + \varphi_{,k}) + (\varphi\theta + \varphi_{,k} u^k)(q_i u_j + q_j u_i) \\
&- 2(\varphi_{,k} q^k + \varphi q^k D u_k)(a_i a_j + \tfrac{1}{4} g_{ij}) \qquad \ldots(2.12)
\end{aligned}
$$

$$
\begin{aligned}
k^{-1} \overset{4}{H}_{ij} &= \Lambda(u_i u_j - \tfrac{1}{4} g_{ij}) + 2\mu_0 u_{(i}\gamma_{j)}^{k} (\varphi D u_k + \varphi_{,k}) \\
&- 2\mu_1 \varphi \sigma_{ij} + 2u_{(i}\gamma_{j)}^{k} \varphi D q_k - 2\mu\varphi h_{(i}\overset{*}{\gamma}{}_{j)}^{k} D h_k \\
&+ 2q_{(i}\hat{\gamma}_{j)}^{k} (\varphi D u_k + \varphi_{,k}) + 2\varphi \{q^k \sigma_{k(i} u_{j)} + q^k \omega_{k(i} u_{j)}\} \\
&- 2\mu\varphi h^k \omega_{k(i} h_{j)} + (\varphi\theta + \varphi_{,k} u^k)(q_i u_j + q_j u_i) \qquad \ldots(2.13)
\end{aligned}
$$

where $\Lambda = -\frac{4}{3}(2\mu_0)^{1/2}\varphi\{(2\mu_0)^{1/2} u^k\}_{;k}$. Similarly other types of choices can be made but we are interested in the analysis of the above mentioned skeleton structures which will be discussed one by one in the form of theorems.

Using relations (2.9) in the resulting equation obtained by the comparison of (2.10) with (2.8), we have

$$h^k \omega_{ki} = 0 \text{ as } \mu \neq 0, \ \varphi \neq 0 \qquad ...(2.14)$$

which gives the conditions for 'restricted steady state' (Esposito and Glass 1977) for the perfect magnetofluid motion in the presence of differential rotation with the magnetic field. It can be shown from source-free part of Maxwell field eqns. (2.9*b*) and (2.14) that $(\frac{1}{2} n^{-1/3} \mu \mid h \mid ^2 u^k)_{;k} = 0$, where n is any scalar function denoting the particle number. This conservation expression is obtained by Esposito and Glass. In view of (2.10), (2.14) and (2.9*a* - *c*) we observe the following theorem :

Theorem 2.1 — An imperfect magnetofluid space-time admits an FCRC quasi-symmetry property (2.10) with $\xi^i = \varphi u^i$ iff (i) $\{\varphi(\mu_0 u^k + q^k)\}_{;k} = 0$ and (ii) $(\frac{1}{2} n^{-1/3} \mu \mid h \mid ^2 u^k)_{;k} = 0$.

Using eqns. (2.9) in the resulting equation obtained by the comparison of (2.11) with (2.8), we get

$$Du_k + (\ln \varphi)_{,k} = 0 \text{ as } \mu_0 \neq 0, \ q_i \neq 0 \qquad ...(2.15)$$

which shows that the acceleration vector is curl-free. Thus we have the following theorem;

Theorem 2.2 — An imperfect magnetofluid space-time admits an FCRC quasi-symmetry property (2.11) with $\xi^i = \varphi u^i$ iff (i) $\{\varphi(\mu_0 u^k + q^k)\}_{;k} = 0$ and (ii) the fluid acceleration is curl-free.

Remark : In general the fluid acceleration is not curl-free in the quotient space.

Using eqns. (2.9) in the resulting equation obtained by the comparison of (2.12) with (2.8), we have

$$2\mu_1 \varphi \sigma_{ij} - 2\varphi \{q^k \sigma_{k(i} u_{j)} + q^k \omega_{k(i} u_{j)}\} - 2\mu\varphi h^k \omega_{k(i} u_{j)} = 0 \qquad ...(2.16)$$

which gives following conditions

$$\sigma_{ij} = 0, \ q^k \omega_{ki} = h^k \omega_{ki} = 0, \text{ as } \mu_1 \neq 0, \ \varphi \neq 0, \ q_k \neq 0 \qquad ...(2.17a)$$

and

$$\sigma_{ij} = \omega_{ij} = 0 \qquad ...(2.17b)$$

which seems to be quite restrictive. Let us investigate about the conditions given by (2.17*a*).

Using Greenberg's (1970) decomposition for the space-like unit heat-flux vector a_i

$$a_{i;j} = \hat{\sigma}_{ij} + \hat{\omega}_{ij} + \hat{\theta}\hat{\gamma}_{ij} - \hat{D}a_i a_j + Da_i u_j - (Da_k u^k) u_i u_j$$

$$+ (\hat{D}a_k u^k) u_i a_j + a_{k;j} u^k u_i, \qquad ...(2.18)$$

where $\hat{\sigma}_{ij}$, $\hat{\omega}_{ij}$, $\hat{\theta}$ denote the shear, rotation and expansion of the heat flux tubes respectively, and (2.17*a*) in the resulting equation obtained by contracting the following relation (Prasad 1978a)

$$q^i = \gamma^i_j (\omega^{jk}_{;k} + \tfrac{2}{3}\theta^{;j} - \sigma^{jk}_{;k}) + (\omega^i_k + \sigma^i_k) Du^k \qquad ...(2.19)$$

with q_i we get

$$| q | = \hat{\omega}_{jk}\omega^{jk} - \tfrac{2}{3}\hat{D}\theta \qquad ...(2.20)$$

which shows that $| q | \neq 0$ since $\theta = \omega^{jk} \neq 0$. Now we may interpret the conditions given by (2.17*a*) as the analogue of conditions characterizing the 'restricted steady state' (Esposito and Glass 1977) for the imperfect magnetofluid motion in the presence of thermal heat. Thus in view of (2.12), (2.17*a*), (2.9*b*) and (2.9*c*) we state the following theorem.

Theorem 2.3 — An imperfect magnetofluid space-time admits an FCRC quasi-symmetry property (2.12) with $\xi^i = \varphi u^i$ iff (i) $\{\varphi(\mu_0 u^k + q^k)\}_{;k} = 0$ and (ii) the imperfect magnetofluid motion is in 'restricted steady state'.

Using relations (2.9) in the resulting equation obtained by the comparison of (2.13) with (2.8), we get

$$\varphi_{,k} q^k + \varphi q^k D q_k = 0 \quad \text{as} \quad (a_i a_j + \tfrac{1}{4} g_{ij}) \neq 0 \qquad ...(2.21)$$

By virtue of (2.5) and (2.21), we obtain

$$(\ln \varphi)_{,k} q^k + | q |^2 / KT = 0 \qquad ...(2.22)$$

which yields the following theorem;

Theorem 2.4 — An imperfect magnetofluid-time admits an FCRC quasi-symmetry property (2.13) with $\xi^i = \varphi u^i$ iff (i) $\{\varphi(\mu_0 u^k + q^k)\}_{;k} = 0$ and (ii) $(\ln \varphi)_{,k} q^k + | q |^2/KT = 0$.

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REFERENCES

Davis, W. R. (1974). Conservation Laws in Einstein's General Theory of Relativity, in Lanczos Festschrift, edited by B. K. P. Scaife. (London and N. Y.) p. 29.

Davis, W. R., Green, L. H., and Norris, L. K. (1976). Relativistic matter fields admitting Ricci collineations and related conservation laws. *Nuovo Cim.*, **34B**, 256.

Eckart, C. (1940). The hydrodynamics of irreversible process II. *Phys. Rev.*, **58**, 919.

Esposito, F. P., and Glass, E. N. (1976). General relativistic magnetofluid. *J. Math. Phys.*, **18**, 705.

Greenberg, P. J. (1970). The general theory of space-like congruence with an application to vorticity in relativistic hydrodynamics. *J. math. Analysis Applic.*, **30**, 128.

Lichnerowicz, A. (1967). Relativistic Hydrodynamics and Magnetohydrodynamics. Benjamin, New York, p. 93.

Norris, L. K., Green, L. H., and Davis, W. R. (1977). Fluid space-times including electromagnetic fields admitting symmetry mappings belonging to the family of contracted Ricci collineations. *J. Math. Phys.*, **18**, 1305.

Oliver, D. R., and Davis, W. R. (1976). Perfect fluids and symmetry mappings leading to conservation laws. *J. Math. Phys.*, **17**, 1790.

Prasad, G. (1978a). Relativistic magnetofluids and symmetry mappings I. *Indian J. pure appl. Math.*, **9**, 682-91.

——————— (1978b). Relativistic magnetofluids and symmetry mappings II. *Indian J. pure appl. Math.*, **9**, 692-701.