CYLINDRICALLY SYMMETRIC SELF-GRAVITATING FLUIDS WITH PRESSURE EQUAL TO ENERGY DENSITY

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Solutions of Einstein's field equations are obtained under the assumption that (1) the source of the gravitational field is a perfect fluid with pressure p equal to energy density ρ, (2) the space time is cylindrically symmetric with two degrees of freedom, and (3) the metric is given by three functions of two variables. The coordinate transformation to comoving coordinate is discussed. The Hawking-Penrose energy conditions and Thorne's C-energy are also studied. Some physically interesting solutions are obtained. The relation of the present work to Einstein-Rosen waves is also investigated.

1. INTRODUCTION

In a recent paper Tabensky and Taub (1973) have found that Einstein's field equations for self-gravitating perfect fluid with pressure p equal to rest energy density ρ and four-velocity $u_{,i}$ are equivalent to the field equations

(a) $R_{ij} = -2\sigma_{,i}\,\sigma_{,j}$

(b) $\Box\, \sigma = (\sqrt{-g}\, \sigma_{,i} g^{ij})_{,j} = 0$ ...(1)

when irrotationality is imposed, viz.

$$u_i = \frac{\sigma_{,i}}{(\sigma_{,k}\sigma^{,k})} \qquad ...(2)$$

The pressure p and energy momentum tensor T_{ij} are related to σ by

$$p = \rho = \sigma_{,k}\sigma^{,k} \qquad ...(3)$$

$$T_{ij} = 2\sigma_{,i}\sigma_{,j} - g_{ij}\sigma_{,k}\sigma^{,k} \qquad ...(4)$$

The units are chosen so that the velocity of light $C = 1$ and Newton's constant of gravitation $G = 1/8\pi$. A comma means partial derivative.

Further Letelier (1975) and Letelier and Tabensky (1975) have obtained cylindrically symmetric solutions of the field eqns. (1). The purpose of this paper is to discuss the solution of eqns. (1) in a cylindrically symmetric space time with two degrees of freedom (Stachel 1966) expressed as

$$ds^2 = e^{2A-2B}(dt^2 - dr^2) - (C^2 e^{2B} + r^2 e^{-2B})\, d\phi^2 - e^{2B} dz^2 - 2Ce^B d\phi dz \qquad \ldots(5)$$

where A, B and C are functions of r and t only, and r, ϕ, z, t correspond respectively to x^1, x^2, x^3, x^4 coordinates. When $C = 0$, the metric (5) reduces to Einstein-Rosen metric (Einstein and Rosen 1937; Rosen 1954) with one degree of freedom.

In Section 2 we find the solution of eqns. (1) for the metric (5). In section 3 the coordinates transformation that enables us to write the solution in comoving coordinates is discussed. In section 4, the Hawking-Penrose energy condition (Hawking and Penrose 1970) are verified. In Section 5, some special solutions corresponding to monochromatic and pulse wave solutions for σ are obtained and Thorne's C-energy is discussed. Also the relation of the present work to Einstein-Rosen waves is pointed out.

2. The Solution of Field Equation

For the metric (5) the field eqns. (1) and the pressure p are

$$B_{11} - B_{44} + \frac{B_1}{r} - \left(\frac{e^{2B}}{2r^2}\right)(C_1^2 - C_4^2) = 0 \qquad \ldots(6)$$

$$C_{11} - C_{44} - \frac{C_1}{r} + 4(B_1 C_1 - B_4 C_4) = 0 \qquad \ldots(7)$$

$$A_1 = r(B_1^2 + B_4^2) + \left(\frac{e^{4B}}{4r}\right)(C_1^2 + C_4^2) + \tfrac{1}{2} r(\sigma_1^2 + \sigma_4^2) \qquad \ldots(8)$$

$$A_4 = 2r_1 B_1 B_4 + \left(\frac{e^{4B}}{2r}\right) C_1 C_4 + r\sigma_1\sigma_4 \qquad \ldots(9)$$

$$A_{11} - A_{44} + B_1^2 - B_4^2 - \left(\frac{e^{4B}}{4r^2}\right)(C_1^2 - C_4^2) = -\tfrac{1}{2}(\sigma_1^2 - \sigma_4^2) \qquad \ldots(10)$$

$$\sigma_{11} - \sigma_{44} + \frac{\sigma_1}{r} = 0 \qquad \ldots(11)$$

$$p = \rho = e^{-2A+2B}(\sigma_4^2 - \sigma_1^2) \qquad \ldots(12)$$

where the indices 1 and 4 indicate partial derivatives with respect to r and t respectively.

Equations (6) and (7) which determine B and C are identical to those of the empty space for the metric (5). Equation (10) can be obtained from (6) – (9) and (11). When B and C are known from (6) and (7) and σ from eqn. (11), eqns. (8) and (9) give A as an integral

$$A = \int \left[\left\{ r(B_1^2 + B_4^2) + \left(\frac{e^{4B}}{4r}\right)(C_1^2 + C_4^2) + \tfrac{1}{2} r(\sigma_1^2 + \sigma_4^2) \right\} dr \right.$$
$$\left. + \left\{ 2rB_1B_4 + \left(\frac{e^{4B}}{2r}\right) C_1C_4 + r\sigma_1\sigma_4 \right\} dt \right]. \qquad \ldots(13)$$

The integrability conditions for A are satisfied by virtue of eqns. (6), (7) and (11). One can always add a constant to A. Further if (g_{ij}, σ) is any solution $(\lambda g_{ij}, \sigma)$ is also a solution whenever λ is a constant. So from now onwards all line elements can be multiplied by a constant conformal factor.

3. Comoving Coordinates

Now we shall discuss how to transform the solution to comoving coordinates which are usually used in hydrodynamics and they are important for physical interpretation.

We can choose σ as the comoving time T. It can be easily seen that the coordinate R defined by

$$dR = r(\sigma_4 dr + \sigma_1 dt) \qquad \ldots(14)$$

and $T = \sigma$ transform the four-velocity u_i to $U_i = (0, 0, 0, U_4)$ and therefore R is comoving. Equation (11) ensures the exactness of the differential (14) defining R.

The required transformation formulae are

$$\left. \begin{array}{ll} T = \sigma(r, t), & R = R(r, t) \\ \Phi = \phi, & Z = z \end{array} \right\} \qquad \ldots(15)$$

where T, R, Φ and Z are comoving coordinates. The Jacobian of (15) is

$$\frac{\partial(R, \Phi, A, T)}{\partial(r, \phi, z, t)} = r(\sigma_4^2 - \sigma_1^2)$$

which can vanish where $p = \rho = 0$ in the nonsingular regions of space time. In comoving coordinates the line element (5) is tranformed to

$$ds^2 = \left\{ \frac{e^{2A-2B}}{(\sigma_4^2 - \sigma_1^2)} \right\} (dT^2 - dR^2) - (C^2 e^{2B} + r^2 e^{-2B})\, d\Phi^2$$
$$- e^{2B} dZ^2 - 2Ce^{2B} d\Phi dZ. \qquad \ldots(16)$$

The line element (16) has a singularity at $r = 0$.

4. The Reality Conditions

In irrotational fluids with the limiting form of the equation of state $p = \rho$, the energy condition $T_{ij}u^i u^j \geqslant 0$ and the Hawking-Penrose condition (Hawking and Penrose 1970)

$$(T_{ij} - \tfrac{1}{2}g_{ij}T)\, u^i u^j \geqslant 0$$

both reduce to

$$\rho = \tfrac{1}{2} e^{-2A+2B} (\sigma_4^2 - \sigma_1^2) \geqslant 0.$$

Thus it is possible that ρ may be negative in some regions of the space-time. The metric does not have necessarily a pathological behaviour when this happens. The way of solving this problem is to fill the region where the energy density is negative with a different kind of fluid, whose energy tensor we prescribe as follows.

From (14) we find that $R_{,i}$ is orthogonal to $\sigma_{,i}$, $\Phi_{,i}$ and $Z_{,i}$. In this region, $R_{,i}$ is a timelike vector and $\sigma_{,i}$ is spacelike. Now let $\hat{R}_{,i}$, $\hat{\Phi}_{,i}$, $\hat{Z}_{,i}$, $\hat{\sigma}_{,i}$ denote the corresponding unit vector fields. If we use the fact that

$$g_{ij} = \hat{R}_{,i}\hat{R}_{,j} - \hat{\sigma}_{,i}\hat{\sigma}_{,j} - \hat{\Phi}_{,i}\hat{\Phi}_{,j} - \hat{Z}_{,i}\hat{Z}_{,j}$$

the stress energy tensor (4) can be written as

$$T_{ij} = (-\,\sigma_{,k}\sigma^{,k})\, [\hat{R}_{,i}\hat{R}_{,j} + \hat{\sigma}_{,i}\hat{\sigma}_{,j} - \hat{\Phi}_{,i}\hat{\Phi}_{,j} - \hat{Z}_{,i}\hat{Z}_{,j}].$$

This stress energy tensor is that of an anisotropic fluid with positive rest energy density $(-\sigma_{,k},\ \sigma^{,k})$ and vanishing heat flow vector. In this case both the reality conditions are satisfied.

5. Some Special Solutions and Thorne's C-energy

Equations (6) - (9) are a set of coupled, second order nonlinear partial differential equations and it is difficult to obtain a general solution of these equations. As eqns. (6) and (7) which determine B and C are same as those in the case of empty space, following Stachel (1966) we try some special solutions. Stachel has mentioned two particular cases (i) $B = 0$ and $B = (\tfrac{1}{4}) \log r + b$, where b is a constant. When $B = 0$, eqns. (6) and (7) lead to $C =$ constant which can be eliminated with the help of a coordinate transformation $z' = z + C\phi$, where C is a constant. When $B = (\tfrac{1}{4}) \log r + b$, from eqns. (6) and (7) it follows that C is a function of $t - r$ or $t + r$, but not their sum, because of the nonlinearity of the equations.

Equation (11) is the Euclidean wave equation in cylindrical coordinates from which σ can be obtained by well known method. A typical solution of this equation may be written in the form

$$\sigma = MJ_0(kr) \cos kt, \qquad \ldots(17)$$

where M and k are constants and $J_0(kr)$ is Bessel's function of first kind and of order zero. As suggested by Weber and Wheeler (1957) a physically more interesting case is that of a pulse formed by linear superposition of monochromatic waves with σ of the form (17). One can superpose such waves with an amplitude factor $M = 2\, Ne^{-ak}$ and thus

$$\sigma = 2N \int_0^\infty e^{-ak} J_0(kr) \cos kt \, dk$$

$$= N\left[\{(a - it)^2 + r^2\}^{-1/2} + \{(a + it)^2 + r^2\}^{-1/2}\right]. \qquad \ldots(18)$$

For monochromatic outgoing waves, we have

$C = C(t - r)$, $B = (\frac{1}{4}) \log r + b$, σ given by (17) and

$$A = \frac{1}{16} \log r - \tfrac{1}{2} e^{4b} \int C^{-2} du$$

$$+ \tfrac{1}{2}(L^2 kr) J_0(kr) J_0''(kr) \cos 2kt$$

$$+ \tfrac{1}{2}(L^2 k^2 r^2) \{[J_0'(kr)]^2 - J_0(kr) J_0''(kr)\} \qquad \ldots(19)$$

where $u \equiv t - r$ and a bar over a function means differentiation with respect to its argument.

For monochromatic incoming waves, we have

$C = C(t + r)$, $B = (\frac{1}{4}) \log r + b$, σ given by (17) and

$$A = \frac{1}{16} \log r + \tfrac{1}{2} e^{4b} \int \bar{C}^2 dv$$

$$+ \tfrac{1}{2}(L^2 kr) J_0(kr) J_0''(kr) \cos 2kt$$

$$+ \tfrac{1}{2}(L^2 k^2 r^2) \{[J_0'(kr)]^2 - J_0(kr) J_0''(kr)\} \qquad \ldots(20)$$

where $v \equiv t + r$.

In the case of the pulse wave also one can write down the expression for A when $B = (\frac{1}{4}) \log r + b$, $C = C(t \pm r)$ and σ is given by (18).

Further Thorne (1965) has given a definition of energy for cylindrically symmetric systems termed as C-energy. His definition has been adapted by one of the present authors (Singh 1977) to cylindrical systems in a scalar-tensor theory.

In this definition of C-energy a quantity $E(r, t)$, expressed in terms of the generators of the system, acts as a potential function from which C-energy flux vector P^i is calculated for the metric (5), the function E is

$$E(r, t) = \left(\frac{1}{4G}\right) A(r, t) = 2A(r, t) \qquad \ldots(21)$$

where we have taken $G = 1/8\pi$, G being the usual universal gravitational constant. Thus the use of the expression for A in (21) will give E consisting of two parts, one corresponding to g_{ij} and the other to σ both contributing positively to the C-energy density.

When $C = 0$, the metric (5) reduces to the Einstein-Rosen metric (1937) (*see* Rosen 1954 also) in which case the field equations have already been investigated by Lal and Singh (1973) and Latelier (1975). The cylindrical gravitational waves are related to a special class of spherical and toroidal waves (Marder 1969, 1972) and therefore the solutions can easily be related to these waves.

Remarks

It is interesting to remark that the solutions found in this paper can be transformed to solutions of Brans-Dicke theory in the vacuum (Dicke 1962).

The solutions can also be interpreted as the solutions of Einstein's equation with a massless scalar field source, since such a source has the same stress-energy tensor as an irrotational fluid with $p = \rho$ (Tabensky and Taub 1973).

REFERENCES

Dicke, R. H. (1962). Mach's principle and invariance under transformations of units. *Phys. Rev.*, **125**, 2163.

Einstein, A., and Rosen, N. (1937). On gravitational waves. *J. Franklin Inst.*, **223**, 43.

Hawking, S. W., and Penrose, R. (1970). The singularities of gravitational collapse and cosmology. *Proc. R. Soc.*, A **314**, 529.

Lal, K. B., and Singh, T. (1973). Cylindrical wave solutions of Einstein's field equations of general relativity containing zero rest-mass scalar field I. *Tensor, N.S.*, **27**, 211.

Letelier, P. S., and Tabensky, R. R. (1975). Self-gravitating fluids with cylindrical symmetry. *J. Math. Phys.*, **16**, 1488.

——— (1975). Cylindrical self gravitating fluids with pressure equal to energy density. *Nuovo Cim.*, **28B**, 407.

Marder, L. (1969). Gravitational waves in general relativity XI : Cylindrical-spherical waves. *Proc. R. Soc.*, A **313**, 83.

——— (1972). Gravitational waves in general relativity. XII. correspondence between toroidal and cylindrical waves. *Proc. R. Soc.*, A **327**, 123.

Rosen, N. (1954). Some cylindrical gravitational waves. *Bull. Res. Council Israel*, **3**, 328.

Singh, T. (1977). An exact solution of a scalar-tensor theory of gravitation. *Proc. Indian Acad. Sci.*, **85A**, 90.

Stachel, J. J. (1966). Cylindrical gravitational news. *J. Math. Phys.*, **7**, 1321.

Tabensky, R., and Taub, A. H. (1973). Plane symmetric self-gravitating fluids with pressure equal to energy density, *Commun. Math. Phys.*, **29**, 61.

Thorne, K. S. (1965). Energy of infinitely long, cylindrically symmetric systems in general relativity. *Phys. Rev.*, **138**, B 251.

Weber, J., and Wheeler, J. A. (1957). Reality of cylindrical gravitational waves of Einstein and Rosen. *Rev. Mod. Phys.*, **29**, 509.