CURVATURE COLLINEATION FOR THE FIELD OF TOTAL RADIATION

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In this paper the cylindrically symmetric field of total radiation, obtained by Krishna Rao, is considered and a particular solution is presented, which admits curvature collineation (CC). The CC vector is found to be proper and the concomitant conservation law for null gravitational fields is satisfied identically.

1. INTRODUCTION

The curvature collineation (CC) symmetry, which was introduced by Katzin *et al.* (1969), is defined by the condition that the V_4 admits a vector ξ^i such that

$$\underset{\xi}{\mathcal{L}} R^h_{ijk} = 0 \qquad \text{...(1.1)}$$

where $\mathcal{L}$ denotes Lie derivative with respect to ξ^i for the coordinates transformation $\bar{x}^i = x^i + \xi^i \delta t$, δt being positive infinitesimal. The investigation of this symmetry was strongly motivated by the all important role of the Riemannian curvature tensor in the theory of general relativity. An important feature of the CC is that it indicates a type of elastic deformation of space-times for which gravitational properties are preserved. Several previously known symmetries, viz., motion, affine collineation etc. are special cases of CC and others are closely related to CC.

In this paper, starting with the general metric, we obtain a cylindrically symmetric field of total radiation which exhibits curvature collineation. Here the CC vector is proper and the conservation law for null gravitational fields is satisfied identically.

2. TOTAL RADIATION FIELD

The Einstein-Rosen cylindrically symmetric metric is

$$ds^2 = e^{(2\gamma-2\psi)}(dt^2 - dr^2) - r^2 e^{-2\psi} d\phi^2 - e^{2\psi} dz^2 \qquad \text{...(2.1)}$$

where γ and ψ are functions of r and t alone. Rao (1964) obtained a particular solution of (2.1), which can be expressed in cylindrical polar coordinates r, ϕ, z and time t as

$$ds^2 = e^{2f}(dt^2 - dr^2) - r^2 d\phi^2 - dz^2 \qquad \text{...(2.2)}$$

where f is a function of the retarded time $u = t - r$. The metric (2.2) is interpreted as representing a non-empty space-time pervaded with gravitational radiation. The Weyl conformal curvature tensor of the space-time is of Petrov type N (Rao 1963).

For the metric (2.2), the non-vanishing components of the Christoffel symbol Γ^i_{jk} are

$$\Gamma^1_{11} = \Gamma^1_{44} = \Gamma^4_{14} = -\Gamma^1_{14} = -\Gamma^4_{11} = -\Gamma^4_{44} = -f',$$

$$\Gamma^1_{22} = -re^{-2f}, \quad \Gamma^2_{12} = \frac{1}{r}, \qquad \ldots(2.3)$$

where "dash" denotes ordinary differentiation with respect to u.

The non-zero components of the Riemann curvature tensor R^h_{ijk} are

$$R^1_{212} = R^1_{224} = R^4_{212} = -R^4_{242} = -re^{-2f}.f',$$

$$R^2_{412} = R^2_{424} = R^2_{121} = -R^2_{124} = -r^{-1}.f'. \qquad \ldots(2.4)$$

The eqn. (1.1) is expressed as

$$\mathcal{L}R^h_{ijk} = R^h_{ijk,m}\,\xi^m - R^m_{ijk}\,\xi^h_{,m} + R^h_{mjk}\,\xi^m_{,i}$$

$$+ R^h_{imk}\,\xi^m_{,j} + R^h_{ijm}\,\xi^m_{,k} = 0 \qquad \ldots(2.5)$$

where a "comma" stands for partial differentiation. From the algebraic symmetries on the indices we find that, in a V_4, eqn. (2.5) formally represents 96 equations. Considering these equations by the use of the metric tensor defined by (2.2), we get the following set of equations :

The non-vanishing components of the Riemann curvature tensor give

$$\mathcal{L}R^1_{212} = 0 \Rightarrow (re^{-2f}.f')'(\xi^1 - \xi^4) + (re^{-2f}.f')\{\xi^1_{,4} + \xi^4_{,1} - 2\xi^2_{,2}\} = 0 \qquad \ldots(2.6)$$

$$\mathcal{L}R^2_{121} = 0 \Rightarrow (r^{-1}f')'(\xi^1 - \xi^4) + (2r^{-1}f')\{\xi^4_{,1} - \xi^1_{,1}\} = 0 \qquad \ldots(2.7)$$

$$\mathcal{L}R^1_{224} = 0 \Rightarrow (re^{-2f}.f')'(\xi^1 - \xi^4) + (re^{-2f}.f')\{\xi^1_{,1} + 2\xi^1_{,4}$$

$$- 2\xi^2_{,2} - \xi^4_{,4}\} = 0 \qquad \ldots(2.8)$$

$$\mathcal{L}R^2_{124} = 0 \Rightarrow (r^{-1}.f')'(\xi^4 - \xi^1) + (r^{-1}.f')\{\xi^1_{,1} + \xi^4_{,4} - \xi^1_{,4} - \xi^4_{,1}\} = 0 \qquad \ldots(2.9)$$

$$\mathcal{L}R^2_{412} = 0 \Rightarrow (r^{-1}.f')'(\xi^1 - \xi^4) + (r^{-1}.f')\{\xi^1_{,4} + \xi^4_{,1} - \xi^1_{,1} - \xi^4_{,4}\} = 0 \quad ...(2.10)$$

$$\mathcal{L}R^4_{212} = 0 \Rightarrow (re^{-2f}.f')'(\xi^1 - \xi^4) + (re^{-2f}.f')\{2\xi^4_{,1} - 2\xi^2_{,2} - \xi^1_{,1} + \xi^4_{,4}\} = 0 \quad ...(2.11)$$

$$\mathcal{L}R^2_{424} = 0 \Rightarrow (r^{-1}.f')'(\xi^1 - \xi^4) + (2r^{-1}.f')\{\xi^1_{,4} - \xi^4_{,4}\} = 0 \quad ...(2.12)$$

$$\mathcal{L}R^4_{242} = 0 \Rightarrow (re^{-2f}.f')'(\xi^4 - \xi^1) + (re^{-2f}.f')\{2\xi^2_{,2} - \xi^1_{,4} - \xi^4_{,1}\} = 0. \quad ...(2.13)$$

The vanishing components of the Riemann curvature tensor provide

$$\mathcal{L}R^1_{213} = 0 \Rightarrow \xi^2_{,3} = 0 \quad ...(2.14)$$

$$\mathcal{L}R^1_{223} = 0 \Rightarrow \xi^1_{,3} - \xi^4_{,3} = 0 \quad ...(2.15)$$

$$\mathcal{L}R^1_{442} = 0 \Rightarrow e^{2f}\xi^1_{,2} - r^2\xi^2_{,4} = 0 \quad ...(2.16)$$

$$\mathcal{L}R^1_{241} = 0 \Rightarrow \xi^2_{,1} + \xi^2_{,4} = 0 \quad ...(2.17)$$

$$\mathcal{L}R^2_{441} = 0 \Rightarrow e^{2f}\xi^2_{,4} - r^2\xi^2_{,1} = 0 \quad ...(2.18)$$

$$\mathcal{L}R^2_{423} = 0 \Rightarrow \xi^4_{,3} = 0 \quad ...(2.19)$$

$$\mathcal{L}R^2_{123} = 0 \Rightarrow \xi^1_{,3} = 0 \quad ...(2.20)$$

$$\mathcal{L}R^3_{442} = 0 \Rightarrow \xi^3_{,2} = 0 \quad ...(2.21)$$

$$\mathcal{L}R^3_{242} = 0 \Rightarrow \xi^3_{,1} + \xi^3_{,4} = 0 \quad ...(2.22)$$

$$\mathcal{L}R^4_{142} = 0 \Rightarrow e^{2f}\xi^4_{,2} + r^2\xi^2_{,1} = 0 \quad ...(2.23)$$

$$\mathcal{L}R^4_{241} = 0 \Rightarrow \xi^2_{,4} + \xi^2_{,1} = 0 \quad ...(2.24)$$

$$\mathcal{L}R^1_{112} = 0 \Rightarrow e^{2f}\xi^1_{,2} + r^2\xi^2_{,1} = 0 \quad ...(2.25)$$

$$\mathcal{L}R^2_{221} = 0 \Rightarrow e^{2f}\xi^1_{,2} + r^2\xi^2_{,1} + e^{2f}\xi^4_{,2} = 0 \quad ...(2.26)$$

$$\mathcal{L}R^{2}_{224} = 0 \Rightarrow r^2(\xi^2_{,1} - \xi^2_{,4}) + e^{2f}(\xi^1_{,2} - \xi^4_{,2}) = 0 \qquad ...(2.27)$$

$$\mathcal{L}R^{4}_{442} = 0 \Rightarrow e^{2f}\xi^4_{,2} + r^2\xi^2_{,4} = 0. \qquad ...(2.28)$$

Here redundant and trivial equations have been omitted.

3. Determination of the CC Vector ξ^i

By inspection we find, from eqns. (2.6) – (2.28), that

$$\xi^1_{,2} = 0, \ \xi^1_{,3} = 0, \ \xi^1_{,4} = 0 \qquad ...(3.1)$$

$$\xi^4_{,1} = 0, \ \xi^4_{,2} = 0, \ \xi^4_{,3} = 0, \ \xi^4_{,4} = \xi^1_{,1} \qquad ...(3.2)$$

$$\xi^3_{,2} = 0, \ \xi^3_{,1} = -\ \xi^3_{,4} \qquad ...(3.3)$$

$$\xi^2_{,1} = 0, \ \xi^2_{,3} = 0, \ \xi^2_{,4} = 0. \qquad ...(3.4)$$

In view of (3.1) and (3.2), the solutions for ξ^1 and ξ^4 can be put as follows

$$\xi^1 = c\,.\,r + \alpha \qquad ...(3.5)$$

$$\xi^4 = c\,.\,t + \beta \qquad ...(3.6)$$

where c, α and β are arbitrary constants. Equation (3.3) gives ξ^3 as

$$\xi^3 = (t - r)\,Z(z) \qquad ...(3.7)$$

$Z(z)$ being an arbitrary function of z.

In view of (3.5), (3.6) and (3.7), eqns. (2.6) – (2.13) give only two independent equations as

$$(re^{-2f}\,.\,f')'\,(\xi^1 - \xi^4) - (re^{-2f}\,.\,f')\,\xi^2_{,2} = 0 \qquad ...(3.8)$$

$$(r^{-1}\,.\,f')'\,(\xi^1 - \xi^4) - 2c\,.\,(r^{-1}\,.\,f') = 0. \qquad ...(3.9)$$

Equation (3.9) can be written as

$$\frac{(r^{-1}\,.\,f')'}{(r^{-1}\,.\,f')} = \frac{2c}{\{(\alpha - \beta) - c\,.\,(t - r)\}}$$

which on integration with respect to u gives

$$f' = \frac{c_1\,.\,r}{\left\{\dfrac{\alpha - \beta}{c} - (t - r)\right\}^2} \qquad ...(3.10)$$

c_1 being an arbitrary constant.

We now consider eqn. (3.8). For the consistency of the equation, we take $\xi^2_{,2} = c$. Then (3.8) can be written in the form

$$\frac{(re^{-2f} . f')'}{(re^{-2f} . f')} = \frac{1}{\left\{\frac{\alpha - \beta}{c} - (t - r)\right\}}$$

which on integration gives

$$f' = \frac{c_2 . e^{2f}}{r\left\{\frac{\alpha - \beta}{c} - (t - r)\right\}}, \qquad ...(3.11)$$

c_2 being another constant of integration.

From (3.10) and (3.11), we obtain

$$e^{2f} = \frac{Ar^2}{\{B - (t - r)\}} \qquad ...(3.12)$$

where

$$A = \frac{c_1}{c_2}, \; B = \frac{(\alpha - \beta)}{c}.$$

Thus the line-element (2.2) becomes

$$ds^2 = \frac{Ar^2}{\{B - (t - r)\}} (dt^2 - dr^2) - r^2 d\phi^2 - dz^2 \qquad ...(3.13)$$

for which the components of the CC vector ξ^i are

$$\left.\begin{aligned} \xi^1 &= c . r + \alpha \\ \xi^2 &= c . \phi + \gamma \\ \xi^3 &= (t - r) . Z(z) \\ \xi^4 &= c . t + \beta \end{aligned}\right\} \qquad ...(3.14)$$

where γ is an arbitrary constant.

Now we shall see that the CC vector ξ^i admitted by the metric (3.13) is proper. A CC which is not degenerate in the sense of being at the same time a stronger symmetry such as a motion or affine collineation etc. is called proper. By the use of (3.14) and the definition $h_{ij} = \xi_{i;j} + \xi_{j;i}$, we see that $h_{12} = r(c\phi + \gamma) \neq 0$; which shows that the CC vector ξ^i does not define motion. For affine collineation ξ^i must satisfy $h_{ij;k} = 0$. It can be seen that $h_{12;1} = r\,(c . \phi + \gamma)\, f'$. Therefore ξ^i is not an affine collineation vector. It can easily be seen that $\xi^i_{;ijk} \neq 0$ in general. So ξ^i does not define either a conformal motion (including homothetic motion) or projective and conformal collineations. Hence the CC vector ξ^i given by (3.14) is proper for the metric (3.13).

4. Conservation Law

The space-time characterized by the metric (3.13) is null in Petrov-Pirani classification scheme. For this metric the scalar curvature R vanishes while $R_{ij} \neq 0$. Katzin *et al.* (1969) have shown that, if a null gravitational field admits a CC, then the field admits a conservation law of the form

$$(\sqrt{-g}\, T^{ijkm}\xi_k)_{;m} = 0 \qquad \ldots(4.1)$$

where T^{ijkm} is the Bel-Robinson tensor defined as

$$T^{ijkm} = 2C^{irjs}\, C^{km}_{\ \ rs} \qquad \ldots(4.2)$$

C_{hijk} being the conformal curvature tensor. Here a "semicolon" stands for covariant differentiation throughout.

The non-vanishing components of the tensor C_{hijk} are

$$\frac{C_{1212}}{r^2} = \frac{C_{1224}}{r^2} = \frac{C_{2424}}{r^2} = -\,C_{3131} = C_{3134} = -\,C_{3434} = \frac{f'}{2r}. \qquad \ldots(4.3)$$

The non-zero components of the tensor T^{ijkm} are

$$T^{1111} = T^{1114} = T^{1414} = T^{4441} = T^{4444} = e^{-8f}\cdot\left(\frac{f'}{r}\right)^2 \qquad \ldots(4.4)$$

where f is given by (3.12).

Now substituting T^{ijkm} from (4.4) in (4.1), it can easily be seen that the conservation law concomitant with the existence of CC vector is satisfied identically.

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