ON DECOMPOSITION OF PRIMARY IDEALS OF Γ-RINGS

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The purpose of this paper is to point out that some of the well-known results on commutative rings hold for commutative Γ-rings and proof can be observed to be parallel to those for the commutative rings.

1. INTRODUCTION

Let $M = \{a, b, c, d, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \delta, \ldots\}$ be additive Abelian groups. If there exists a mapping $M \times \Gamma \times M \rightarrow M$ satisfying the conditions

1. $(a + b)\,\alpha c = a\alpha c + b\alpha c$

   $a(\alpha + \beta)\,b = a\alpha b + a\beta b$

   $a\alpha(b + c) = a\alpha b + a\alpha c;$

2. $(a\alpha b)\,\beta c = a\alpha(b\beta c);$

3. $(a\alpha b) = 0$ and if $a, b \neq 0$ then $\alpha = 0;$

then M is said a Γ-ring (cf. Barnes 1966). Also, if there is a mapping $\Gamma \times M \times \Gamma \rightarrow \Gamma$ satisfying the conditions

1a. same as 1 above;

2a. $(a\alpha b)\,\beta c = a(\alpha b\beta)\,c = a\alpha(b\beta c);$

3a. $a\alpha b = 0$, if $\alpha \neq 0$ then either $a = 0$ or $b = 0$;

then M is said to be a Γ-ring in the sense of Nobusawa (1964). A Nobusawa Γ-ring M is said to be commutative if $a\alpha b = b\alpha a$ and $\alpha a\beta = \beta a\alpha$ hold for every $a, b \in M$ and $\alpha, \beta \in \Gamma$.

The notion of Γ-ring was introduced by Nobusawa (1964) and he used it to generalize the Wedderburn theorem. Later on, it was discussed by several authors (Luh 1968, 1969; Barnes 1966; Coppage and Luh 1971).

Throughout the paper we have assumed that Γ-ring is in the sense of Nobusawa and commutative. All the basic notations follow from Barnes (1966) and Coppage and Luh (1971). If A is an ideal of a Γ-ring M and B is the subset of Γ-ring M then by the term $A : B$ we mean the set of all those elements x of M such that $x\alpha b \in A$ for every $\alpha \in \Gamma$.

2. Prime Ideal and Radical

Definition — An ideal P of a Γ-ring M is said to be prime if for ideals A and B, $A \,\Gamma\, B \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$.

Definition — By the radical of an ideal A, we mean the set of all the elements $a \in M$ such that $(a\alpha)^n a \in A$ for some positive integer n and for every $\alpha \in \Gamma$.

Some of the propositions given below are deduced by us. We do not add the proofs of the Propositions 2.1 – 2.3 here because their proofs are similar to that of commutative ring (*see* Zariski and Samuel 1958). However, the Propositions 2.4 – 2.7 are proved here.

Proposition 2.1 — Let P be a prime ideal of a Γ-ring M and the product $a_1 \Gamma a_2 \Gamma \ldots \Gamma a_n$ is contained in P then at least one $a_i \in P$.

Proposition 2.2 — Let P be a prime ideal of *a* Γ-ring M and $A_1 \Gamma A_2 \Gamma \ldots \Gamma A_n$ is contained in P then for some i, A_i is contained in P.

Proposition 2.3 — If A is an ideal of a Γ-ring M then $M \,\Gamma\, \text{Rad } A \subseteq \text{Rad } A$.

Proposition 2.4 — Radical of an ideal A is an ideal containing A.

Proof : Suppose $a, b \in \text{Rad } A$ and $\alpha \in \Gamma$, $(a\alpha)^n a \in A$, $(b\alpha)^m b \in A$. Since $\{(a+b)\alpha\}^{m+n}(a+b)$ can be written as sum of the terms of the form $(a\alpha)^{m+n} a$, $(b\alpha)^{m+n} b$, $(a\alpha)^p (b\alpha)^q b$ and $(b\alpha)^p (a\alpha)^q a$, for some p and q such that $p+q = m+n$. For such p and q, $(a\alpha)^p (b\alpha)^q b$ and $(b\alpha)^p (a\alpha)^q a \in A$. Hence $\{(a+b)\alpha\}^{m+n}(a+b) \in A$ for some positive integer $m+n$. Thus $a+b \in \text{Rad } A$. From the proposition (2.3) we have $M \,\Gamma\, \text{Rad } A \subseteq \text{Rad } A$. Thus Rad A is an ideal containing A.

Proposition 2.5 — A is an ideal of a Γ-ring M then $\text{Rad Rad } A = \text{Rad } A$.

Proof : Since $A \subseteq \text{Rad } A \Rightarrow \text{Rad } A \subseteq \text{Rad Rad } A$. We have only to show that $\text{Rad Rad } A \subseteq \text{Rad } A$. For this, we consider $a \in \text{Rad Rad } A$, then $(a\alpha)^m a \in \text{Rad } A$, for some positive integer m, and also $\{[(a\alpha)^m a]\,\alpha\}^n [(a\alpha)^m a] \in A$. It can be easily verified that $\{[(a\alpha)^m a]\,\alpha\}^n [(a\alpha)^m a] = (a\alpha)^{mn+m} a \in A$ imply $a \in \text{Rad } A$. Thus $\text{Rad Rad } A = \text{Rad } A$.

Proposition 2.6 — Let $A_1, A_2, \ldots, A_n$ be the ideals of a Γ-ring M, then $\text{Rad }(A_1 \Gamma A_2 \Gamma \ldots \Gamma A_n) = \text{Rad }(A_1 \cap A_2 \cap \ldots \cap A_n)$.

Proof : Proof of this proposition is straightforward.

Proposition 2.7 — Radical of an ideal A is the intersection of all prime ideals containing A.

Proof : Suppose that A is an ideal of a Γ-ring M and P is any prime ideal containing A. Let $a \in \text{Rad } A$, then for some positive integer m and every $\alpha \in \Gamma$, we have $(a\alpha)^m a \in A \subseteq P$. From Proposition 2.1, $a \in P$. Thus the radical of A is contained in the intersection of all its prime ideals. To prove the converse, we have to show, $b \notin \text{Rad } A \Rightarrow b \notin p$ for some prime ideal of A. Given $b \notin \text{Rad } A$, we must construct a prime P containing A such that $b \notin P$ (cf. McCoy 1964).

3. Primary Ideal of Γ-Ring

Definition — A proper ideal Q of Γ-ring M is said to be primary if for each $\alpha \in \Gamma$, $x\alpha y \in Q$ and $y \notin Q$ follows, $(x\alpha)^m x \in Q$ for some positive integer m. Also if Rad $Q = P$ then Q is said to be P-primary.

Proposition 3.1 — Radical of a primary ideal Q is prime.

Proof: Let for some x, y; $x\alpha y \in \text{Rad}(Q) = P$ for all $\alpha \in \Gamma$, such that $x \notin P$. Then for some $\beta \in \Gamma$, $(x\beta)^m x \notin Q$ for every m. However $x\alpha y \in P$ implies $[(x\alpha y)\beta]^m (x\alpha y) \in Q$ for some m i.e. $[(x\beta)^m x]\,\alpha\,[(y\alpha)^m y] \in Q$. This in turn gives $([(y\alpha)^m y]\,\alpha)^n (y\alpha)^m y \in Q$ for some n. Thus $(y\alpha)^{mn+m+n} y \in Q$ and hence $y \in P$. This completes the proof.

Proposition 3.2 — Let A be a primary ideal of a Γ-ring M and S be an arbitrary non-empty subset of M, then $A : S$ is primary ideal. Further if $S \not\subseteq \text{Rad } A$ then $A : S = A$.

Proof: Let $x\alpha y \in A : S$ and $y \notin A : S$. Thus for some $\beta \in \Gamma$ and $s \in S$, $(y\beta)\, s \notin A$. However $(x\alpha y)\,\beta S \in A$, and A is primary. Hence $(x\alpha)^m x \in A$ for some m. Then obviously $A \subseteq A : S$ gives $(x\alpha)^m x \in A : S$. This proves $A : S$ is primary. Second part is trivial.

4. Primary Decomposition of Ideals

Definition — An ideal A possesses a primary decomposition, if it can be expressed in the form $A = Q_1 \cap Q_2 \cap \ldots \cap Q_n$, where each Q_i is primary. A primary decomposition $A = Q_1 \cap Q_2 \cap \ldots \cap Q_n$ is said to be reduced or irredundant if $\bigcap_{j \neq i} Q_j \not\subseteq Q_i$ for all i.

Definition — An irredundant primary decomposition $K = N_1 \cap N_2 \cap \ldots \cap N_t$ is said to be normal decomposition if all the prime ideals P_i corresponding to the primary components N_i are distinct.

Definition — A Γ-ring M is said to be Neotherian if for every ascending sequence

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots \subseteq A_n \subseteq \ldots$$

of ideals, there exists an integer n such that $A_n = A_{n+p}$, $P = 0, 1, 2, \ldots$.

Definition — An ideal is said to be reducible if $A = A_1 \cap A_2$, where A_1 and A_2 are ideals properly containing A, otherwise irreducible.

Lemma 4.1 — In a Neotherian Γ-ring every ideal is a finite intersection of irreducible ideals.

Lemma 4.2 — In a Neotherian Γ-ring M every irreducible ideal is primary.

PROOF : Let A be an ideal of a Γ-ring M, suppose that it is not primary i.e. there exist elements $x, y \in M$ not in A such that $x\alpha y \in A$, $y \notin A$ and $(x\alpha)^n x \notin A$, for all positive integers n. Consider the ascending sequence of ideals of Γ-ring M.

$$A : x \subseteq A : x\alpha x \subseteq A : (x\alpha)^2 x \subseteq \ldots \subseteq A : (x\alpha)^n x \subseteq A : (x\alpha)^{n+1} x \subseteq \ldots .$$

Let n be the smallest positive integer n such that $A : (x\alpha)^n x = A : (x\alpha)^{n+p} x$, $p = 0, 1, 2, 3, \ldots$. Now we claim that

$$A = (A + M\,\Gamma(x\alpha)^n x) \cap (A + (y)).$$

It is obvious that $A \subseteq (A + M\,\Gamma(x\alpha)^n x) \cap (A + (y))$.

Conversely if we consider

$$d \in (A + M\,\Gamma(x\alpha)^n x) \cap (A + (y))$$

we have

$$d = u + \Sigma\, a_i\alpha_i(x\alpha)^n x = v + (my + \Sigma\, b_j\beta_j y),$$

$(u, v \in A\,;\ a_i, b_j \in M\,;\ \alpha_i, \beta_j \in \Gamma\,;$ and $m \in Z)$.

Since $x\alpha y \in A$ we have

$$x\alpha d = x\alpha u + \Sigma\, x\alpha a_i\alpha_i(x\alpha)^n x = x\alpha v + x\alpha(my) + \sum_j x\alpha b_j\beta_j y \in A.$$

$$\Rightarrow x\alpha u + \Sigma\, x\alpha a_i\alpha_i(x\alpha)^n x \in A$$

$$\Rightarrow \sum_i x\alpha a_i\alpha_i(x\alpha)^n x \in A$$

$$\Rightarrow \sum_i a_i\alpha_i(x\alpha)^{n+1} x \in A \ \text{ or } \ \sum_i a_i\alpha_i(x\alpha)^n x \in A$$

$$\Rightarrow d = u + \Sigma\, a_i\alpha_i(x\alpha)^n x \in A.$$

Thus $(A + M\Gamma(x\alpha)^n x) \cap (A + (y)) \subseteq A$. Hence A is reducible.

Theorem 4.3 — In a Neotherian Γ-ring M every ideal is a finite intersection of primary ideals.

PROOF : Proof of this theorem follows directly from Lemmas 4.1 and 4.2.

Proposition 4.4 — Let $K = N_1 \cap N_2 \cap \ldots \cap N_s$, where N_i is P_i-primary ideals of Γ-ring M, G be an arbitrary sub Γ-ring M and let P be a prime ideal of Γ-ring M. Then Rad $(K : G) = P_1 \cap P_2 \cap \ldots \cap P_s$ and P contains $K : G$ if it contains at least one of $P_1, P_2, \ldots, P_s$.

PROOF : $K = N_1 \cap N_2 \cap \ldots \cap N_s$. It is easy to verify that $K : G = (N_1 : G) \cap (N_2 : G) \cap \ldots \cap (N_s : G)$. Since $P_i = \text{Rad}\,(N_i : G)$. It follows from Proposition 3.3 that $K : G \subseteq \text{Rad}\,(K : G) = P_1 \cap P_2 \cap \ldots \cap P_s$. If P contains one of P_i then evidently it contains $K : G$.

Proposition 4.5 — Let $K = N_1 \cap N_2 \cap \ldots \cap N_m$, where N_i is a P_i primary ideal of a Γ-ring M and let S be any ideal such that $S \not\subseteq P_1$, $S \not\subseteq P_2, \ldots, S \not\subseteq P_t$ and also $S \subseteq P_{t+1}$, $S \subseteq P_{t+2}, \ldots, S \subseteq P_m$. Then $K : S = N_1 \cap N_2 \cap \ldots \cap N_t$.

PROOF : $K : S = (N_1 \cap N_2 \cap \ldots \cap N_m) : S$

$$= \bigcap_{i=1}^{m} (N_i : S)$$

$$= N_1 \cap N_2 \cap \ldots \cap N_t, \text{ using Proposition 3.2.}$$

This proves the result.

Proposition 4.6 — Let $K = N_1 \cap N_2 \cap \ldots \cap N_m$ and $K = N'_1 \cap N'_2 \cap \ldots \cap N'_n$ be the two normal decomposition of K, an ideal of a Γ-ring M. Where $\text{Rad}\, N_i = P_i$, $\text{Rad}\, N'_j = P'_j$ then $n = m$ and the two sets $\{P_1, P_2, P_3, \ldots, P_m\}$ and $\{P'_1, P'_2, P'_3, \ldots, P'_n\}$ are the same though the order may be different.

PROOF : In order to prove the theorem, we have only to show that any ideal in $\{P_1, P_2, \ldots, P_m\}$ is an ideal in $\{P'_1, P'_2, \ldots, P'_n\}$.

Let P be an ideal in $\{P_1, P_2, \ldots, P_m\}$ without the loss of the generality we may assume that $P = P_t$ for some number $(1 \leqslant t \leqslant m)$ such that $P \not\subseteq P_1, P \not\subseteq P_2, \ldots, P \not\subseteq P_{t-1}$, $P = P_t$, $P \subseteq P_{t+1}$, $P \subseteq P_{t+2}, \ldots, P \subseteq P_m$. Thus from the Proposition 4.5 we have

$$K : P = N_1 \cap N_2 \cap \ldots \cap N_{t-1}.$$

To prove the contradiction, we suppose that

$P \not\subseteq \{P'_1, P'_2, P'_3, \ldots, P'_n\}$. Thus from the Proposition (3.2) we have.

$$K : P = (N'_1 \cap N'_2 \cap \ldots \cap N'_n) : P$$

$$= \bigcap_{i=1}^{n} (N'_i : P)$$

$$= N'_1 \cap N'_2 \cap \ldots \cap N'_n = K.$$

Thus $K = N_1 \cap N_2 \cap \ldots \cap N_{t-1}$. This is the contradiction to the fact that

$$K = N_1 \cap N_2 \cap \ldots \cap N_m.$$

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