ON SOME COMMUTATOR IDENTITIES IN FINITE GROUPS

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In this paper we shall prove that if A, B are subgroups of a finite group G that satisfy $[a, b, a] = 1 = [b, a, b]$ for all a in A and b in B, then $[A, B, B, B]^4 = 1$. We shall deduce from this that if A or B is a subgroup of odd order of a finite group G then $[A, B, B, B] = 1$ and that the second Engel group is nilpotent of class atmost three.

For elements a, b in a group G we write $a^b = b^{-1}ab$. The commutator of a and b is defined as $[a, b] = a^{-1}b^{-1}ab$ and $[[a, b], c] = [a, b, c]$ (Hall 1959). Throughout this paper we shall assume $a, a_1, a_2, \ldots, \in A$ and $b, b_1, b_2, \ldots, \in B$ and the identities $[a, b, a] = 1 = [b, a, b]$ are satisfied for all a in A and b in B. All groups considered will be finite.

Lemma 1 — (i) $[a^b, a] = 1$;

(ii) $[a^{b_1}, a^{b_2}] = 1$.

Lemma 2 — (i) $[a^{b_1}, b_2, b_2] = 1$;

(ii) $[a \,.\, a^{b_1}, b_2, b_2] = 1$;

(iii) $[a^n \,.\, a^{b_1}, b_2, b_2] = 1$ for every positive integer n.

PROOF : Proof for (i) follows very easily.

For (ii), consider

$$x = [a \,.\, a^{b_1}, b_2, b_2]$$

then

$$\begin{aligned} x &= [[a, b_2]^{a^{b_1}} \,.\, [a^{b_1}, b_2], b_2] \\ &= [[a, b_2]^{a^{b_1}}, b_2]^{[a^{b_1}, b_2]} \, [[a^{b_1}, b_2], b_2] \\ &= [a^{b_1}, b_2]^{-1} \, [[a, b_2]^{a^{b_1}}, b_2] \, [a^{b_1}, b_2]^{b_2} \end{aligned}$$

or

$$x^{[a^{b_1}, b_2]^{-1}} = [[a, b_2]^{a^{b_1}}, b_2] [b_2, [a^{b_1}, b_2]^{-1}].$$

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But

$$
\begin{aligned}
[a, b_2]^{a^{b_1}} &= a^{-b_1}\, a^{-1}a^{b_2}\, a^{b_1} \\
&= a^{-b_1}\, a^{-1}a^{b_1}\, a^{b_2} \\
&= [a^{b_1}, a]\, a^{-1}a^{b_2} \\
&= [a, b_2]
\end{aligned}
$$

and

$$
\begin{aligned}
[b_2, [a^{b_1}, b_2]^{-1}] &= [b_2, [a^{b_1}, b_2^{-1}]^{b_2}] \\
&= [[a^{b_1}, b_2^{-1}]^{b_2}, b_2^{-1}]^{b_2} \\
&= [a^{b_1 b_2}, b_2^{-1}, b_2^{-1}]^{b_2} \\
&= 1.
\end{aligned}
$$

Hence

$$x^{[a^{b_1}, b_2]^{-1}} = 1$$

which gives $x = 1$.

Proof of (iii) follows by a simple induction on n.

Lemma 3 — (i) $[a, b_1, b_2] = [a, b_2, b_1]^{-1}$;

(ii) $[a, b_1, b_2, b_3] = [a, b_1, b_3, b_2]^{-1}$.

PROOF: (i) $[b, a, b] = 1$ implies

$$[b, a] = [b, a]^b$$

or $\quad [a, b^{-1}] = [b, a]$

or $\quad a^{-1}a^{b^{-1}} = a^{-b}a$

or $\quad a^{-1}a^{b^{-1}} = a\,a^{-b}$

or $\quad a^{b^{-1}} = a^2a^{-b}$.

Let $\quad x = [a, b_1, b_2]\,[a, b_2, b_1]$

then

$$
\begin{aligned}
x &= [a^{-1}a^{b_1}, b_2]\,[a^{-1}a^{b_2}, b_1] \\
&= a^{-b_1}\, a\, a^{-b_2}\, a^{b_1 b_2}\, a^{-b_2}\, a\, a^{-b_1}\, a^{b_2 b_1} \\
&= a^2 a^{-2b_1}\, a^{-2b_2}\, a^{b_1 b_2}\, a^{b_2 b_1}
\end{aligned}
$$

(*equation continued on p.* 920)

$$= (a^2a^{-b_1})^2 \, (a^2a^{-b_1})^{-b_2} \, a^{-2}a^{b_2b_1}$$

$$= a^{2b_1^{-1}} \, a^{-b_1^{-1}b_2} \, a^{-2}a^{b_2b_1}$$

$$= a^{b_1^{-1}b_2^{-1}} \, a^{-2}a^{b_2b_1}.$$

Therefore

$$x^{b_2} = a^{b_1^{-1}} \, a^{-2b_2} \, a^{b_2b_1b_2}$$

$$= a^{-b_2} \, a^{b_1^{-1}} \, a^{-b_2} \, a^{b_2b_1b_2}$$

$$= [a^{-b_1^{-1}} \, a^{b_2}, \, b_1b_2]$$

$$= [a^{b_1^{-1}}, \, b_1b_2, \, b_1b_2]$$

$$= 1.$$

Hence, $x = 1$

i.e., $[a, b_1, b_2] = [a, b_2, b_1]^{-1}.$

Proof of (ii) is similar.

Lemma 4 — $[a, [b_1, b_2]] = [a, b_1, b_2]^2.$

PROOF : We have by Lemma 3,

$$a^{b_2b_1} = a^{-2}a^{2b_1} \, a^{2b_2} \, a^{-b_1b_2}.$$

Since Lemma 3 is true if we replace a by any of its conjugate, replacing a by $a^{b_1^{-1}}$ and b_2 by b_2^{-1} we get

$$a^{b_1^{-1}b_2^{-1}b_1} = a^{-2b_1^{-1}} \, a^2a^{2b_1^{-1}b_2^{-1}} \, a^{-b_2^{-1}}$$

or $$a^{[b_1, b_2]} = a^{b_1^{-1}b_2^{-1}b_1b_2} = a^{-2b_1^{-1}b_2} \, a^{2b_2} \, a^{2b_1^{-1}} \, a^{-1}.$$

Now

$$[a, [b_1, b_2]] = a^{-1} \, a^{[b_1, b_2]}$$

$$= a^{-1} \, a^{-2b_1^{-1}b_2} \, a^{2b_2} \, a^{2b_1^{-1}} \, a^{-1}$$

$$= a^{-2} \, a^{2b_2} \, a^{2b_1^{-1}} \, a^{-2b_1^{-1}b_2}$$

$$= a^{-2}a^{2b_2} \, a^4 \, a^{-2b_1}(a^{-4} \, a^{2b_1})^{b_2}$$

$$= a^{-2} \, a^{2b_2} \, a^4 \, a^{-2b_1} \, a^{-4b_2} \, a^{2b_1b_2}$$

(*continued on p.* 921)

$$= (a^{-b_1} a a^{-b_2} a^{b_1 b_2})^2$$
$$= [a^{-1} a^{b_1}, b_2]^2$$
$$= [a, b_1, b_2]^2.$$

Lemma 5 — $[[a, b_1], [b_2, b_3]] = [a, b_1, b_2, b_3]^2$.

PROOF: From Lemma 4, we have

$$[a^{-1}, [b_2, b_3]] = [a^{-1}, b_2, b_3]^2$$

and also

$$[a^{b_1}, [b_2, b_3]] = [a^{b_1}, b_2, b_3]^2$$

we get

$$[a^{-1} a^{b_1}, [b_2, b_3]] = [a^{-1} a^{b_1}, b_2, b_3]^2$$

i.e.,

$$[[a, b_1], [b_2, b_3]] = [a, b_1, b_2, b_3]^2.$$

Lemma 6 — $[a, [[b_1, b_2], b_3]] = 1$.

PROOF: We have

$$\begin{aligned}
[a, [b_1, b_2], b_3] &= [[a, b_1, b_2]^2, b_3] \\
&= [a, b_1, b_2, b_3]^2 \\
&= [[a, b_1], [b_2, b_3]] \\
&= [a, [b_2, b_3], b_1]^{-1} \\
&= [[a, b_2, b_3]^2, b_1]^{-1} \\
&= [a, b_2, b_3, b_1]^{-2} \\
&= [[a, b_2], [b_3, b_1]]^{-1} \\
&= [a, [b_3, b_1], b_2] \\
&= [[a, b_3, b_1]^2, b_2] \\
&= [a, b_3, b_1, b_2]^2 \\
&= [[a, b_3], [b_1, b_2]]
\end{aligned}$$

which gives

$$a^{b_3 [b_1, b_2]} = a^{[b_1, b_2] b_3}$$

and hence

$$[a, [[b_1, b_2], b_3]] = 1.$$

Now we state and prove our main theorem.

Theorem 1 — Let A and B be two subgroups of a finite group G that satisfy $[a, b, a] = 1 = [b, a, b]$ for all a in A and b in B, then $[A, B, B, B]^4 = 1$ (Mehdi 1977).

PROOF : Consider

$$\begin{aligned} 1 &= [a, [[b_1, b_2], b_3]] \\ &= [a, [b_1, b_2], b_3]^2 \\ &= [[a, b_3], [b_1, b_2]]^{-2} \\ &= [a, b_3, b_1, b_2]^{-4}. \end{aligned}$$

This gives that

$$[a, b_3, b_1, b_2]^4 = 1$$

and hence

$$[A, B, B, B]^4 = 1.$$

This completes the proof of the theorem.

This theorem implies the following Corollaries (Levi 1942) :

Corollary 1 — If A or B is a subgroup of odd order of finite group G, then $[A, B, B, B] = 1$.

Corollary 2 — If G is a group of odd order then the second Engel group is nilpotent of class atmost three.

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REFERENCES

Hall, M. (1959). The Theory of Groups. Macmillan Company, New York.

Levi, F. W. (1942). Groups in which commutator operation satisfies certain algebraic conditions. *J. Indian math. Soc., N.S.*, **6**, 87-97.

Mehdi, J. (1977). *C*-Commutations in groups. Ph.D. thesis, AMU, Aligarh.