ON SOME GENERATING FUNCTIONS FOR THE GENERALIZED HYPERGEOMETRIC POLYNOMIALS

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In this paper, authors introduce a new generalized hypergeometric polynomial and then establish a generating relation for the generalized hypergeometric polynomial so defined. Certain particular cases have also been discussed.

1. INTRODUCTION

During the course of an attempt to unify and extend the study of well-known sets of polynomials, we have found a generalized hypergeometric polynomial in the form

$$K_n(x) = (\alpha)_n/n!\; x^{n(k-1)}\; {}_{p+k+l}F_{q+k+l}\left[\begin{matrix}\Delta(-n, k),\ \Delta(n+\alpha, l),\ a_1, \ldots, a_p; \\ \Delta(\alpha, k+l), \qquad b_1, \ldots, b_q;\end{matrix}\; uk^k l^l x^\mu\right]$$

$$= (\alpha)_n/n! \sum_{r=0}^{\infty} \frac{\prod_{i=0}^{k-1}\left(\frac{-n+i}{k}\right)_r \prod_{i=0}^{l-1}\left(\frac{n+\alpha+i}{l}\right)_r (a_p)_r\, u^r k^{kr} l^{lr}}{r! \prod_{j=0}^{k+l-1}\left(\frac{\alpha+i}{k+l}\right)_r (b_q)_r}$$

$$\times\; x^{\mu r + n(k-1)} \qquad \ldots(1.1)$$

where n, k, l and α are non-negative integers, the parameters a's and b's are independent of n, $\Delta(\alpha, k)$ is denoted by

$$\alpha/k,\ (\alpha+1)/k,\ \ldots,\ (\alpha+k-1)/k,$$

and

$$(a_p)_r = \prod_{i=1}^{p} (a_i)_r;\quad (b_q)_r = \prod_{j=1}^{q} (b_j)_r.$$

The polynomial (1.1) is in a generalized form and therefore yields the number of known and unknown polynomials by particular choice of parameters.

2. Generating Function

Let $G(y)$, analytic at $y = 0$, have the expansion

$$G(y) = \sum_{n=0}^{\infty} h_n y^n,$$

and $g_n(x)$ be defined by the relation

$$(1 - t)^{-\alpha} x^{n(k-1)} G\,[u(k + l)^{k+l} (- t)^k x^{\mu+k(k-1)}/(1 - t)^{k+l}]$$

$$= \sum_{n=0}^{\infty} g_n(x)\, t^n. \qquad \ldots(2.1)$$

For sufficiently small values of t, the left-hand side of (2.1) can be expanded in an absolutely convergent double series and the terms can be arranged so as to have a convergent power series in t. Thus, using the known results

$$\left.\begin{aligned} &(a)_{n+k} = (a + n)_k\, (a)_n,\ (1 - x)^{-c} = \sum_{n=0}^{\infty} (c)_n\, x^n/n! \\ &(k - n)! = (-1)^n k!/(-k)_n,\ (a)_{nk} = k^{nk} \prod_{i=0}^{k-1} \left(\frac{a + i}{k}\right)_n \\ &\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/\lambda]} A(k, n - \lambda k) \end{aligned}\right\} \qquad \ldots(2.2)$$

where integer $\lambda \geqslant 0$ and [ ] is our customary greatest integer symbol, we can easily have

$$g_n(x) = \sum_{r=0}^{[n/k]} (\alpha)_n/n!\; x^{n(k-1)} \frac{\prod_{i=0}^{k-1} \left(\frac{-n + i}{k}\right)_r \prod_{i=0}^{l-1} \left(\frac{n + \alpha + i}{l}\right)_r}{\prod_{j=0}^{k+l-1} \left(\frac{\alpha + j}{k + l}\right)_r}$$

$$\times\, h_r u^r k^{kr} l^{lr} x^{\mu r}. \qquad \ldots(2.3)$$

Taking

$$G(y) = {}_pF_q \left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix}\; u(k + l)^{k+l} (- t)^k x^{\mu+k(k-1)}/(1 - t)^{k+l}\right]$$

whose conditions for convergence are well known, the $g_n(x)$ defined by (2.3) becomes (1.1).

Therefore, using (2.1), we have the generating function

$$(1-t)^{-\alpha}\, x^{n(k-1)}\, {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix}\; u(k+l)^{k+l}(-t)^k x^{\mu+k(k-1)}/(1-t)^{k+l}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^{n(k-1)}$$

$$\times\, {}_{p+k+l}F_{q+k+l}\left[\begin{matrix} \triangle(-n, k), \triangle(n+\alpha, l), a_1, \ldots, a_p; \\ \triangle(\alpha, k+l), \qquad b_1, \ldots, b_q; \end{matrix}\; uk^k l^l x^{\mu}\right] t^n. \qquad \ldots(2.4)$$

3. Particular Cases

The generating function (2.4) includes a number of particular cases, e.g.,

(i) When $\mu = k = l = u = 1,\ \alpha = 1 + a + b,$ we have

$$(1-t)^{-1-a-b}\, {}_pF_q\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix}\; -4tx/(1-t)^2\right]$$

$$= \sum_{n=0}^{\infty} (1+a+b)_n/n!$$

$$\times\, {}_{p+2}F_{q+2}\left[\begin{matrix} -n, n+a+b+1, \qquad a_1, \ldots, a_p; \\ (1+a+b)/2, (2+a+b)/2, b_1, \ldots, b_q; \end{matrix}\; x\right] t^n$$

$$= \sum_{n=0}^{\infty} f_n^{(a,b)}\left[\begin{matrix} a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix}\; x\right] t^n \qquad \ldots(3.1)$$

a generating function for a new generalization of Sister Celin's polynomial and reduces to that of Sister Celine's polynomial [Rainville 1960, p. 290, (1)] on putting $a = b = 0$.

(ii) Setting $p = 3,\ q = 2,\ \mu = k = l = u = 1,\ \alpha = 1 + a + b,\ a_1 = (1 + a + b)/2,$ $a_2 = (2 + a + b)/2,\ a_3 = \rho,\ b_1 = 1 + a,\ b_2 = \nu,$ we obtain

$$(1-t)^{-1-a-b}\,{}_3F_2\left[\begin{matrix}(1+a+b)/2,\ (2+a+b)/2,\ \rho; \\ 1+a, \qquad \nu;\end{matrix}\ -4tx/(1-t)^2\right]$$

$$=\sum_{n=0}^{\infty}(1+a+b)_n/(1+a)_n\,H_n^{(a,b)}(\rho,\nu,x)\,t^n \qquad ...(3.2)$$

a generating function for the generalized Rice's polynomial introduced by Khandekar (1964, p. 158). Also (3.2) is the same relation (4.2) due to Khandekar (1964) and reduces to eqn. (4) of Rainville (1960, p. 288) for the Rice's polynomial on putting $a = b = 0$.

(iii) Relation (3.2) yields, on taking $\rho = \nu$, the generating function [Rainville 1960, p. 261, (1)] for the Jacobi polynomial.

(iv) The substitutions $p = q = \mu = k = u = 1$, $l = 0$, $a_1 = \alpha$, $b_1 = 1 + a$ give, from (2.4) the generating function [Rainville 1960, p. 202, (3)] for the Laguerre polynomial which further reduces to the relation (4) of Rainville (1960, p. 202) on replacing α by $1 + a$.

(v) Taking $p = q = \mu = k = l = u = 1$, $\alpha = 2c$, $a_1 = c$, $b_1 = 1 + b$, we obtain a generating relation for one of the generalizations (Rainville 1953) of the Bessel polynomials

$$(1-t)^{-2c}\,{}_1F_1\,[\,c;\ 1+b;\ -4tx/(1-t)^2]$$

$$=\sum_{n=0}^{\infty}(2c)_n/n!\,{}_2F_2\left[\begin{matrix}-n,\ n+2c\ ; \\ c+1/2,\ 1+b;\end{matrix}\ x\right]t^n \qquad ...(3.3)$$

which reduces to the relation [Rainville 1960, p. 285, (3)] for the Bateman's polynomials on putting $b = 0$ and $c = 1/2$.

(vi) Putting $p = 2$, $q = 0$, $\mu = k = l = 1$, $u = -1/b$, $\alpha = a - 1$, $a_1 = (a-1)/2$, $a_2 = a/2$ in (2.4), we get the generating function for the generalized Bessel polynomials of Krall and Frink [1949, p. 109, (36)]

$$(1-t)^{1-a}\,{}_2F_0\,[(a-1)/2,\ a/2;\ -;\ 4tx/[b(1-t)^2]]$$

$$=\sum_{n=0}^{\infty}(a-1)_n/n!\,y_n(x,a,b)\,t^n \qquad ...(3.4)$$

which on replacing a by $c + 1$ and b by $- 1$, we obtain the generating function [Rainville 1960, p. 294, (6)] for the polynomial $\phi_n(c, x)$ [Rainville 1960, p. 294, (3)].

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