PSEUDO PROJECTIVE AND PSEUDO INJECTIVE MODULES

by A. K. TIWARY *and* B. M. PANDEYA, *Department of Mathematics, Banaras Hindu University, Varanasi* 221005

(*Received* 20 *January* 1978; *after revision* 20 *April* 1978)

The object of this paper is to generalize the concept of quasi projective modules to pseudo projective modules and to study their properties. The following results have been obtained :

(1) A characterization of semi-simple rings in terms of pseudo projectivity.

(2) If a module has a quasi projective cover, then it has a pseudo projective cover.

(3) If every module has a pseudo projective cover, then it has a projective cover.

(4) The following conditions are equivalent :

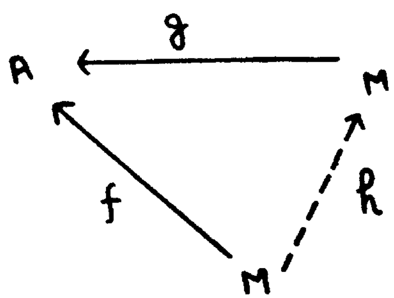
(a) R is left perfect; (b) every module has a pseudo projective cover; (c) every flat module is pseudo projective.

(5) Characterization of semi-simple rings in terms of pseudo injectivity.

INTRODUCTION

The notion of pseudo projective cover of a module is introduced as a natural generalization of that of quasi projective cover defined by Wu and Jans (1967). Characterizations of semi-simple rings by using pseudo projectivity and that of commutative semi-simple ring by using pseudo injectivity have been provided (Singh and Jain 1967). It has also been proved that if every module has a pseudo projective cover then it also has a quasi projective cover and hence a projective cover.

Definition 1.1 — An R-module M is said to be pseudo projective if for any given R-module A, epimorphisms $g : M \to A$ and $f : M \to A$ there exists a homomorphism $h : M \to M$ such that the following diagram is commutative, i.e. $f = g \circ h$.



Remark 1.2 — By a semi-simple ring we mean a ring with unit satisfying descending chain condition such that the lattice of its left ideals is complemented.

Lemma 1.3 — A sufficient condition for an exact sequence

$$0 \to K \to P \to Q \to 0$$

to split is that $P \oplus Q$ is pseudo projective.

PROOF : Proof is on the same lines as in Golan (1970, Lemma 1.1)

Lemma 1.3 enables us to give a more general version of Theorem 1.3 of Golan (1970) as follows :

Theorem 1.4 — For a ring R the following conditions are equivalent :

(1) The direct sum of pseudo projective modules is pseudo projective.

(2) Every pseudo projective module is projective.

(3) $R \oplus M$ is pseudo projective for every simple R-module M.

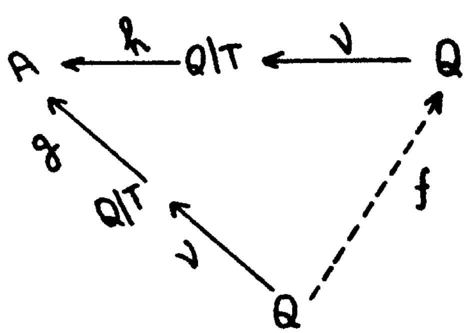
(4) R is semi-simple.

PROOF : See Theorem 1.3 of Golan (1970)

Definition 1.4 — A submodule T of Q is said to be pseudo stable if whenever for $g, h : Q \to A$ epimorphisms such that $T \subseteq \ker g \cap \ker h$. There exists $f \in \text{End } Q$ with $g = h \circ f$, then $f(T) \subseteq T$.

Lemma 1.5 — Let Q be a pseudo projective module and T pseudo stable submodule of Q then Q/T is pseudo projective.

PROOF : Let $g, h : Q/T \to A$ be epimorphisms and $\nu : Q \to Q/T$ natural epimorphism, then since Q is pseudo projective there exists a homomorphism $f \in \text{End } (Q)$ such that the following diagram is commutative.



Thus $g \circ \nu = h \circ \nu \circ f'$.

Also $T \subseteq \ker g \circ \nu \cap \ker h \circ \nu$. Hence by pseudo stability of T in Q, $f(T) \subseteq T$. This implies that $\nu \circ f(T) = 0$, so that $T \subseteq \ker \nu \circ f$.

By homomorphism decomposition theorem, there exists $k \in \text{End}\,(Q/T)$ such that

$$\nu \circ f = k \circ \nu$$

$$\Rightarrow h \circ \nu \circ f = h \circ k \circ \nu$$

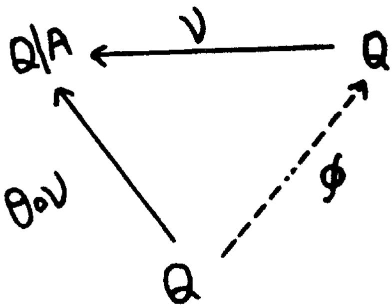
$$\Rightarrow g \circ \nu = h \circ k \circ \nu$$

$$\Rightarrow g = h \circ k$$

Q/T is pseudo projective.

Lemma 1.6 — Quotient of a pseudo stable submodule of a quasi projective module is a pseudo stable submodule. Specifically if T is pseudo stable submodule of a quasi projective module and $A \not\subseteq T$ submodule, then T/A is a pseudo stable submodule of Q/A.

Proof : Let $\lambda, \mu : Q/A \rightarrow B$ be epimorphisms with $T/A \subseteq \ker \lambda \cap \ker \mu$ such that there exists $\theta \in \text{End}\,(Q/A)$ satisfying $\lambda = \mu \circ \theta$. Let $\nu : Q \rightarrow Q/A$ be the natural epimorphism, then since Q is quasi projective, therefore there exists a homomorphism $\phi \in \text{End}\,(Q)$ such that



$$\theta \circ \nu = \nu \circ \phi$$

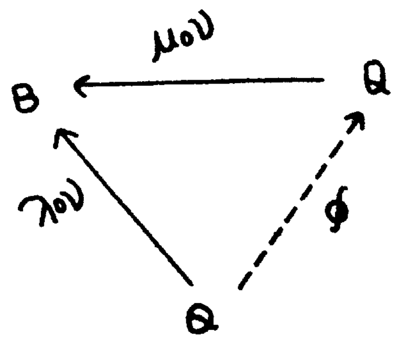
$$\Rightarrow \lambda \circ \nu = \mu \circ \theta \circ \nu = \mu \circ \nu \circ \phi.$$

This implies that the following diagram is commutative.

Also $\qquad \lambda \circ \nu(T) = \lambda(T/A) = 0 \qquad$ ...(i)

and

$$\mu \circ \nu(T) = \mu(T/A) = 0 \qquad \text{...(ii)}$$



(i) and (ii) imply that

$$T \subseteq \ker \lambda \circ \nu \cap \ker \mu \circ \nu$$

Hence $\phi(T) \subseteq T$.

It follows that

$$\theta(T/A) = \theta \circ \nu(T) = \nu \circ \phi(T) \subseteq \nu(T) = T/A.$$

Thus $T/A \subseteq Q/A$ is a pseudo stable submodule as desired.

Lemma 1.7 — Let Q be a quasi projective module and let T be a pseudo stable submodule of Q. If $C \supset T$ is not a pseudo stable submodule of Q then $C/T \subset Q/T$ is not pseudo stable.

PROOF : C not pseudo stable in Q implies that there exists $f \in \text{End}\,(Q)$ $g, h : Q \rightarrow A$ epimorphism, $C \subseteq \ker g \cap \ker h$ with $h = g \circ f$ such that $f(C_0) \notin C$ for some $C_0 \in C$.

Let $\nu : Q \rightarrow Q/T$ natural homomorphism.

Define $\theta : Q/T \rightarrow Q/T$ by

$$\theta(q + T) = f(q) + T$$

i.e. $\theta \circ \nu = \nu \circ f$

θ is clear by a homomorphism and

$$\theta(C_0 + T) = f(C_0) + T \notin Q/T.$$

$T \subseteq \ker g \cap \ker h$ implies that there exists G and H in Hom $(Q/T, A)$, such that

$$g = G \circ \nu \text{ and } h = H \circ \nu.$$

Since g and h are epimorphisms, so are G and H. Now,

$$h = g \circ f \Rightarrow H \circ \nu = G \circ \nu \circ f = G \circ \theta \circ \nu$$

$$\Rightarrow H = G \circ \theta.$$

Thus there exists $\theta \in \text{End}\,(Q/T)$ with $H = G \circ \theta$

$$H(C/T) = H \circ \nu(C) = h(C) = 0.$$

Similarly $\quad G(C/T) = 0$

so that

$$C/T \subseteq \ker H \cap \ker G.$$

But

$$\theta(C_0 + T) \nsubseteq C/T.$$

Hence C/T is not pseudo stable.

Definition 1.8 — A pseudo projective module S is said to be a pseudo projective cover of a module M if

(i) there exists a minimal epimorphism $\psi : S \to M$

(ii) $\ker \psi$ does not contain any non-zero pseudo stable submodule of S.

Proposition 1.9 — If a module has a quasi projective cover, then it has a pseudo projective cover.

PROOF : Let $\phi : Q \to M$ be a quasi projective cover of M. Let T be a maximal pseudo stable submodule of Q contained in $\ker \phi$. Then by the Lemma 1.5, Q/T is pseudo projective $T \subseteq \ker \phi$ implies there exists $\psi : Q/T \to M$ such that $\phi = \psi \circ \nu$ where $\nu : Q \to Q/T$ is natural homomorphism onto implies ψ is onto and $\ker \psi = \ker \phi/T$

Also

$$A/T + \ker \psi = Q/T$$

$$\Rightarrow A + \ker \phi = Q$$

$$\Rightarrow A = Q$$

$$\Rightarrow A/T = Q/T$$

$\ker \psi \subseteq Q/T$ is small submodule.

Let, if possible $C/T \subseteq \ker \psi$ be a non-zero pseudo stable submodule of Q/T.

Then ker $\phi \supset C \supset T$. C is not pseudo stable in Q since in side ker ϕ, T is maximal submodule. Hence C/T is not pseudo stable by Lemma 1.7. Thus $\psi : Q/T \to M$ is a pseudo projective cover of M.

Remark 1.10 — Koehler (1966) has proved in Theorem 1.1 that if every module has a quasi projective cover. Then every module has a projective cover. We show that the result is true in more general case where quasi projective cover is replaced by pseudo projective cover. The proof of the following theorem which is almost exactly the same as given by Koehler (1966), is presented just to show that it works in case of pseudo projective covers.

Theorem 1.11 — If every module has a pseudo projective cover, then it has a projective cover.

PROOF : Consider any module M. There exists a free module F having a basis X of the same cardinality as that of M. Then the one-one mapping of X onto M extends to an epimorphism $\psi : F \to M$. By the hypothesis, $F \oplus M$ has a pseudo projective cover S with minimal epimorphism $\phi : S \to F \oplus M$. Denote the first and second projection of ϕ by p_1 and p_2 respectively. Then $S \xrightarrow{\phi} F \oplus M \xrightarrow{p_1} F$ is onto and so $S/\bar{M} \cong F$ where $\bar{M}$ is ker $p_1 \circ \phi$. By projectivity of F, $S = \bar{M} + T$ (direct) where $S/\bar{M} \cong T$. We can, therefore, express $S = F + \bar{M}$ (direct) after identifying T with F. Let i_1, i_2 denote the natural injections $F \to S$ and $\bar{M} \to S$ respectively. Since $p_2 \circ \phi : S \to M$ is an epimorphism. Let $f = (p_2 \circ \phi)/\bar{M} : \bar{M} \to M$ is as well an epimorphism. To show that ker f is small submodule of $\bar{M}$, let

$$A + \ker f = \bar{M}$$

then

$$A + \ker p_2 \circ \phi = \bar{M}$$

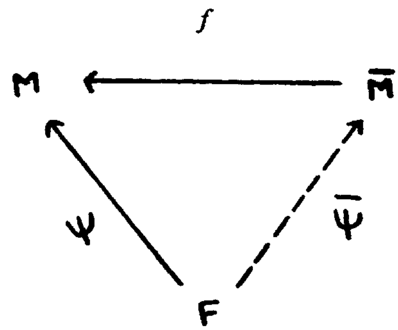
$$\Rightarrow A + \ker \phi + F = F + \bar{M} = S$$

$$\Rightarrow A + F = F + \bar{M}$$

$$\Rightarrow A = \bar{M}.$$

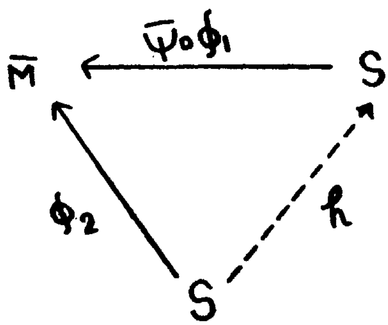
Hence ker f is a small submodule of $\bar{M}$.

Now, by projectivity of F, there exists $\bar{\psi} \in \mathrm{Hom}(F, M)$ such that the following diagram is commutative, i.e. $\psi = f \circ \bar{\psi}$



ker $f \subseteq \bar{M}$ small implies $\bar{\psi}$ is onto.

Using pseudo projectivity of $S = F + \bar{M}$, we can find homomorphism $h : S \rightarrow S$ which makes the following diagram commutative



where $\phi_1 = pr_1\phi$ and $\phi_2 = pr_2\phi$ are the projection mappings. Now, for any $u \in \bar{M}$,

$$u = \phi_2 \circ i_2(u) = \bar{\psi} \circ \phi_1 \circ h \circ i_2(u)$$

whence

$$\bar{M} \xrightarrow{\phi_1 \circ h \circ i_2} F \xrightarrow{\bar{\psi}} \bar{M}$$

is identity on $\bar{M}$.

In view of the epimorphism $\bar{\psi} : F \rightarrow \bar{M}$ which is direct, $\bar{M}$ is a direct summand of F and hence it is projective. $\bar{M}$ is thus the required projective cover of M.

Corollary 1.12 — If every module has a quasi projective cover then it has a projective cover (Golan 1970, Theorem 1.1).

PROOF : The result follows immediately from Proposition 1.9 and Theorem 1.11.

Corollary 1.13 — If every module has pseudo projective cover then R is left perfect.

Corollary 1.14 — If every finitely generated module has a pseudo projective cover then R is left semi perfect.

Definition 1.15 — A ring H is said to be left perfect if every left R-module has a projective cover. One of the several equivalent conditions characterizing left perfect rings given by Bass (1960) in Theorem P is as follows :

Every flat module is projective

Theorem 1.16 — The following conditions are equivalent :

(1) R is left perfect.

(2) Every module has a pseudo projective cover.

(3) Every flat module is pseudo projective.

PROOF : (1) $\Rightarrow$ (2). If R is left perfect, then every module has a projective cover so every module has a quasi projective cover and therefore from Proposition 1.9 every module has a pseudo projective cover.

(2) $\Rightarrow$ (1) : If every module has a pseudo projective cover then every module has a projective cover follows from Theorem 1.11, so R is left perfect.

(1) $\Rightarrow$ (3). If R is left perfect then every flat module is projective from Bass (1960, Theorem P).

So it is pseudo projective.

(3) $\Rightarrow$ (1). To prove that R is left perfect, it suffices to prove that every flat module is projective.

Let M be a flat module. Let F be a free module with basis M. Then F is flat and $F \oplus M$ is flat follows from (Lambek 1966, Prop. 2, p. 133).

Therefore $F \oplus M$ is pseudo projective. Since M is homomorphic image of F, by Lemma 1.3.

M is isomorphic to a direct summand of F and hence M is projective. This completes the proof.

2. PSEUDO INJECTIVE MODULES

Theorem 2.1 — For a commutative ring R the following conditions are equivalent :

(1) The direct sum of every two pseudo injective module is pseudo injective.

(2) Every pseudo injective module is injective.

(3) R is semi-simple.

This theorem is an analogue in pseudo injective case of the corresponding theorem (Koehler 1966, Theorem 2.2) given for quasi injective module. The proof in the latter case holds here also with the change pseudo injective for quasi injective.

Corollary 2.2 — For a commutative ring R the following conditions are equivalent :

(1) The direct sum of every two quasi injective modules is quasi injective.

(2) Every quasi injective module is injective.

(3) R is semi-simple.

PROOF : The result follows from the above theorem and the fact that every quasi injective module is pseudo injective.

Recall that a ring R is called a left V-ring if every simple left R-module is injective.

The following theorem is due to Villiamayor (*see* Faith 1966).

Theorem 2.3 — The following are equivalent :

(1) R is V ring.

(2) Every left ideal of R is the intersection of maximal left ideals which contain it.

(3) For every left R-module M Rad $M = 0$.

Remark 2.4 — In Theorem 2.1, (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) is true without R being commutative.

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