SOME PROBLEMS IN FLEXURE

by U. A. SASTRY, *Department of Mathematics, Regional Engineering College, Warangal* 506004 (*A.P.*)

(*Received* 17 *February* 1978)

In this paper Sokolnikoff's method is used and the flexure problem is reduced to the determination of a single flexure function which agrees with the results of Milne-Thomson and Deutsch.

INTRODUCTION

The flexure problem has been investigated by Sokolnikoff, Stevenson, Morris, Wigglesworth and others. Stevenson reduced the flexure problem to the determination of six canonical flexure functions. Milne-Thomson (1959) has presented a uniform method depending on a special formulation of the boundary condition which leads via conformal mapping of the section on to a unit circle and Cauchy's formulae to a single flexure function. Milne-Thomson's method has been extended to doubly connected sections which can be mapped on a circular annulus by Deutsch (1961).

Here Sokolnikoff's (1956) method has been used and the flexure problem has been reduced to the determination of a single flexure function which agrees with the result of Milne-Thomson (1959) and Deutsch (1961).

FORMULATION OF THE PROBLEM

Consider a cantilever beam of uniform cross-section fixed at the end $Z = 0$ and the other end $Z = 1$ loaded by some distribution of forces statically equivalent to a single force $(W_x, W_y, 0)$ lying in the plane $Z = 1$ and acting at the load point $(x_0, y_0, 1)$. The Z-axis is taken along the central line of the beam while x, y axes are any orthogonal axes intersecting at the centroid of the end $Z = 0$. The lateral surface of the beam is free from external forces and the body forces are absent. Following semi-inverse method of Saint-Venant we put

$$\tau_{xx} = \tau_{xy} = \tau_{yy} = 0. \qquad \ldots(1)$$

The functions τ_{zx}, τ_{zy} and τ_{zz} will be so taken that they satisfy the equations of equilibrium, compatibility equations and boundary conditions. In any section Z units distant from the fixed end we have the relation

$$M_y = W_x(1 - Z) \qquad \ldots(2)$$

so that the stress distribution due to W_x alone in the section would have to be statically equivalent to the moment M_y and to the resultant force W_x. For the problem of bending of beams by couples applied at the ends we get

$$\tau_{zz} = -\frac{M_y x}{I_y} \qquad \text{...(3)}$$

where $I_y = \iint x^2\,dx\,dy$. Now we try to satisfy the conditions of the present problem by taking

$$\tau_{zz} = -E(1-Z)(k_x x + k_y y) \qquad \text{...(4)}$$

where the constants k_x and k_y are to be evaluated from the conditions

$$W_x = \iint \tau_{zx}\,dx\,dy,\quad W_y = \iint \tau_{yz}\,dx\,dy. \qquad \text{...(5)}$$

Using (1) and (4) in the equations of equilibrium we find

$$\left.\begin{aligned} &\frac{\partial \tau_{xz}}{\partial Z} = 0,\ \frac{\partial \tau_{yz}}{\partial Z} = 0 \\ &\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + E(k_x x + k_y y) = 0 \end{aligned}\right\} \qquad \text{...(6)}$$

$$E = 2\mu(1+\sigma).$$

From this it follows that the shear components τ_{xz} and τ_{yz} are functions of (x, y) and have the same value in all sections and

$$\tau_{xz} = \frac{\partial F}{\partial y} - \frac{E k_x x^2}{2},\quad \tau_{yz} = -\frac{\partial F}{\partial x} - \frac{E k_y y^2}{2}. \qquad \text{...(7)}$$

Also the compatibility equations reduce to the form

$$\left.\begin{aligned} &\nabla^2 \tau_{xz} + \frac{E k_x}{(1+\sigma)} = 0 \\ &\nabla^2 \tau_{yz} + \frac{E k_y}{(1+\sigma)} = 0. \end{aligned}\right\} \qquad \text{...(8)}$$

From (7) and (8) we find

$$\left.\begin{aligned} &\frac{\partial(\nabla^2 F)}{\partial y} = 2\mu\sigma k_x \\ &\frac{\partial(\nabla^2 F)}{\partial x} = -2\mu\sigma k_y. \end{aligned}\right\} \qquad \text{...(9)}$$

The solution of (9) can be taken in the form

$$\nabla^2 F = -2\mu\sigma k_y x + 2\mu\sigma k_x y - 2\mu\alpha. \qquad \text{...(10)}$$

Writing

$$z = x + iy,\ \bar{z} = x - iy,\ \beta = k_x + ik_y,\ \bar{\beta} = k_x - ik_y \qquad \text{...(11)}$$

eqn. (10) becomes

$$4\frac{\partial^2 F}{\partial z \partial \bar{z}} = -2\mu\alpha - i\mu\sigma(z\bar{\beta} - \bar{z}\beta). \qquad \text{...(12)}$$

Integrating the above equation we obtain

$$F = \phi(z) + \overline{\phi(z)} - \frac{\mu\alpha z\bar{z}}{2} - \frac{i\mu\sigma}{8}(\bar{\beta}z^2\bar{z} - \beta\bar{z}^2 z). \qquad \text{...(13)}$$

Multiplying the second equation of (7) by i and subtracting from the first equation we find

$$(\tau_{xz} - i\tau_{yz}) = 2i\frac{\partial F}{\partial z} - E(k_x x^2 - ik_y y^2)/2. \qquad \text{...(14)}$$

Using (11) we find

$$(\tau_{xz} - i\tau_{yz}) = 2i\frac{\partial F}{\partial z} - E[\beta(z^2 + \bar{z}^2) + 2\bar{\beta}z\bar{z}]/8. \qquad \text{...(15)}$$

Using (13) in the formula (15) we get

$$(\tau_{xz} - i\tau_{yz}) = \left[2i\frac{\partial \phi}{\partial z} - \mu(1+\sigma)\alpha z^2/4\right] - i\mu\alpha\bar{z}$$
$$+ \mu[-\beta(1+2\sigma)\bar{z}^2/4 - \bar{\beta}z\bar{z}/2] \qquad \text{...(16)}$$

writing $p = i\alpha,\ q = \beta(1+2\sigma)/4,\ r = \bar{\beta}/2$

$$\mu\Phi(z) = 2i\frac{\partial \phi}{\partial z} - \mu(1+\sigma)\beta z^2/4 \qquad \text{...(17)}$$

eqn. (16) can be put in the convenient form

$$(\tau_{xz} - i\tau_{yz}) = \mu[\Phi(z) - p\bar{z} - q\bar{z}^2 - rz\bar{z}]. \qquad \text{...(18)}$$

Formula (18) agrees with that of Milne-Thomson. $\Phi(z)$ is the required flexure function which is analytic over the whole cross-section. Hence the solution of the flexure problem is reduced to the determination of the flexure function.

Boundary Conditions

The lateral surface of the beam is free from external forces is expressed by the condition

$$(\tau_{xz} - i\tau_{yz})\frac{dz}{ds} - (\tau_{xz} + i\tau_{yz})\frac{d\bar{z}}{ds} = 0. \qquad \text{...(19)}$$

Using (18) in (19) we find

$$\Phi(z)\frac{dz}{ds} - \overline{\Phi(z)}\frac{d\bar{z}}{ds} = (p\bar{z} + q\bar{z}^2 + r\bar{z}z)\frac{dz}{ds}$$

$$- (\bar{p}z + \bar{q}z^2 + \bar{r}z\bar{z})\frac{d\bar{z}}{ds}. \qquad \text{...(20)}$$

Formula (20) agrees with the result given by Milne-Thomson.

LOCAL TWIST AT A POINT (x, y) OF A SECTION

Each element of area of a section is rotated in its plane through an angle

$$2w = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \qquad \text{...(21)}$$

The rate of change of this rotation in the Z-direction can be written as

$$\begin{aligned} \partial(2w)\partial Z &= \partial(\partial v/\partial x - \partial u/\partial y)/\partial Z \\ &= \partial(\partial v/\partial Z + \partial w/\partial y)/\partial x - \partial(\partial u/\partial Z + \partial w/\partial x)/\partial y \\ &= \frac{1}{\mu}[\partial \tau_{yz}/\partial x - \partial \tau_{xz}/\partial y]. \end{aligned} \qquad \text{...(22)}$$

An elementary solution of (10) can be taken in the form

$$F = f(x, y) - \mu\sigma(k_y x^3 - k_x y^3)/3 - \mu\alpha(x^2 + y^2)/2 \qquad \text{...(23)}$$

where f is a harmonic function in the section. From (23) and (7) we also find

$$\left.\begin{aligned} \tau_{xz} &= \frac{\partial f}{\partial y} + \mu\sigma k_x y^2 - \mu\alpha y - Ek_x\, x^2/2 \\ \tau_{yz} &= -\frac{\partial f}{\partial x} + \mu\sigma k_y x^2 + \mu\alpha x - Ek_y\, y^2/2. \end{aligned}\right\} \qquad \text{...(24)}$$

From (24) and (22) it follows that

$$\frac{\partial w}{\partial Z} = \alpha + \sigma(k_y x - k_x y). \qquad \text{...(25)}$$

In terms of complex variables the above equation can be written as

$$\frac{\partial w}{\partial Z} = \alpha + \sigma i(\bar{\beta} z - \beta\bar{z})/2 \qquad \text{...(26)}$$

so the mean value of the local twist over the section is α and the terms in (24) that involve α represent a twist of the beam.

DETERMINATION OF THE CONSTANTS k_x AND k_y

Since the resultant of the stresses xz acting over any section must be equal to the component W_x of the applied load we have

$$W_x = \iint \tau_{xz}\, dx\, dy. \qquad \ldots(27a)$$

Similarly

$$W_y = \iint \tau_{yz}\, dx\, dy. \qquad \ldots(27b)$$

Using (24) in (27) we obtain after simplification

$$\left.\begin{aligned} Ek_x &= \frac{(I_x W_x - I_{xy} W_y)}{(I_x I_y - I_{xy}^2)} \\ Ek_y &= \frac{(I_y W_y - I_{xy} W_x)}{(I_x I_y - I_{xy}^2)} \end{aligned}\right\} \qquad \ldots(28)$$

where

$$\left.\begin{aligned} I_x &= \iint y^2\, dx\, dy,\ I_y = \iint x^2\, dx\, dy \\ I_{xy} &= \iint xy\, dx\, dy. \end{aligned}\right\} \qquad \ldots(29)$$

We can easily show that

$$\left.\begin{aligned} M_x &= \iint y\tau_{zz}\, dx\, dy = -(1 - Z)\, W_y \\ M_y &= \iint -x\tau_{zz}\, dx\, dy = (1 - Z)\, W_x \end{aligned}\right\} \qquad \ldots(30)$$

which are the bending moments produced in the section $Z =$ const. by the forces W_x and W_y. The stress distribution over any section is statically equivalent to the load $(W_x, W_y, 0)$ given by

$$\iint \tau_{zx}\, dx\, dy = W_x, \quad \iint \tau_{zy}\, dx\, dy = W_y, \quad \iint \tau_{zz}\, dx\, dy = 0. \qquad \ldots(31)$$

The first two are satisfied by the choice of the constants k_x and k_y while the third follows from the assumption that Z-axis passes through the centroid of the section.

The constant α for shear stresses is determined by the condition that the twisting moment be such that

$$\iint (x\tau_{yz} - y\tau_{xz})\, dx\, dy = (x_0 W_y - y_0 W_x) \qquad \ldots(32)$$

where (x_0, y_0) are the coordinates of the load point relative to any set of axes intersecting at the centroid of the section.

Centre of Flexure

Suppose a load W is applied at $(x_0, y_0, 1)$ and it can be replaced by an equal load at the centre of flexure and by a couple producing the twist α. This gives the following equations.

$$(x_{cf} W_y - y_{cf} W_x) = \iint (x\tau_{yz} - y\tau_{zx})\, dx\, dy \quad (\text{for } \alpha = 0) \qquad \ldots(33)$$

and

$$(x_0 - x_{cf})\, W_y - (y_0 - y_{cf})\, W_x = \iint (x\tau_{yz} - y\tau_{zx})\, dx\, dy \qquad \ldots(34)$$

(with $k_x = k_y = 0$).

We can show that

$$(x\tau_{yz} - y\tau_{zx}) = \tfrac{1}{2}\,[z(\tau_{xz} - i\,\tau_{yz}) - \bar{z}(\tau_{xz} + i\,\tau_{yz})]. \qquad \ldots(35)$$

Using (35) and (18) in (33) and (34) we find

$$x_{cf} W_y - y_{cf} W_x = \frac{\mu i}{2} \iint [z(\bar{\Phi} - \bar{q}z^2 - \bar{r}z\bar{z}) - \bar{z}(\Phi - q\bar{z}^2 - rz\bar{z})]\, dx\, dy \qquad \ldots(36)$$

$$(x_0 - x_{cf})\, W_y - (y_0 - y_{cf})\, W_x = \frac{\mu i}{2} \iint z(\Phi - i\alpha\bar{z}) - \bar{z}(\bar{\Phi} + i\alpha z)]\, dx\, dy. \qquad \ldots(37)$$

Determination of Flexure Function

Let $\quad z = w(\zeta) \qquad \ldots(38)$

be the mapping function which maps conformally the section on the unit circle in the ζ-plane then we have

$$\Phi(z) = \Phi(w(\zeta)) = \psi(\zeta)$$

$$z = w(\sigma),\ \bar{z} = \overline{w(\sigma)},\ \frac{\partial z}{\partial s} = w'(\sigma)\,\frac{d\sigma}{ds} \text{ on the boundary.} \qquad \ldots(39)$$

Using (39) in (20) and multiplying the resulting equation by $\dfrac{d\sigma}{2\pi i(\sigma - \zeta)}$ where ζ is a point inside the unit circle, we get

$$\psi(\zeta)\, w'(\zeta) = \frac{1}{2\pi i} \int H(\sigma)\, \frac{d\sigma}{(\sigma - \zeta)} \qquad \ldots(40)$$

where

$$H(\sigma) = (p + q\bar{w} + rw)\, \bar{w}w' + (\bar{p} + \bar{q}w + \bar{r}\bar{w})\, w\bar{w}'/\sigma^2. \qquad \ldots(41)$$

Simply Connected Region

In this case we use the formula

$$z = w = \sum_{0}^{\infty} b_n \zeta^n \qquad \ldots(42)$$

$\zeta = \sigma$ on the boundary of the unit circle, then we can show that

$$\left.\begin{aligned} \bar{w}w' &= \sum_{-\infty}^{\infty} c_n\sigma^n, \quad c_n = \sum_{-\infty}^{\infty} (n + r + 1)\, b_{n+r+1}\bar{b}_r \\ \bar{w}^2w' &= \sum_{-\infty}^{\infty} d_n\sigma^n, \quad d_n = \sum_{-\infty}^{\infty} c_{n+r}\bar{b}_r \\ w\bar{w}w' &= \sum_{-\infty}^{\infty} e_n\sigma^n, \quad e_n = \sum_{-\infty}^{\infty} b_{-r}c_{n+r}. \end{aligned}\right\} \quad \ldots(43)$$

From (4), (41) and (43) we find

$$\psi(\zeta)\, w'(\zeta) = p \sum_0^{\infty} c_n\zeta^n + q \sum_0^{\infty} d_n\zeta^n + r \sum_0^{\infty} e_n\zeta^n$$
$$+ \bar{p} \sum_{n=2}^{\infty} \bar{c}_{-n}\zeta^{-n+2} + \bar{q} \sum_{n=2}^{\infty} \bar{d}_{-n}\zeta^{-n+2} + \bar{r} \sum_{n=2}^{\infty} \bar{e}_{-n}\zeta^{-n+2} \quad \ldots(44)$$

$\psi(\zeta)$ is the required flexure function.

Cross-section a Cardioid

In this problem we use the transformation formula

$$z = w(\zeta) = c(1 + 2\zeta + \zeta^2) \quad \ldots(45)$$

and we find

$$\left.\begin{aligned} &b_0 = c,\ b_1 = 2c,\ b_2 = c \\ &c_0 = 6c^2,\ c_1 = 2c^2,\ c_2 = c_3 = 0,\ c_{-1} = 6c^2,\ c_{-2} = 2c^2 \\ &d_0 = 10c^3,\ d_1 = 2c^3,\ d_{-1} = 20c^3,\ d_{-2} = 20c^3,\ d_{-3} = 10c^3,\ d_{-4} = 2c^3 \\ &e_0 = 20c^3,\ e_1 = 20c^3,\ e_2 = 10c^3,\ e_3 = 2c^3,\ e_{-1} = 10c^3,\ e_{-2} = 2c^3. \end{aligned}\right\} \ldots(46)$$

Using (46) in (44) we get

$$\frac{\psi(\zeta)\,(1 + \zeta)}{\zeta} = [p(3 + \zeta) + qc(5 + \zeta) + rc(10 + 10\zeta + 5\zeta^2 + \zeta^3)$$
$$+ \bar{p} + \bar{q}c(10 + 5\zeta + \zeta^2) + \bar{r}c] \quad \ldots(47)$$

which agrees with the result of Milne-Thomson.

Doubly Connected Regions

In this case we use the formula

$$z = w(\zeta) = \sum_{-\infty}^{\infty} a_n\zeta^n. \quad \ldots(48)$$

Writing $\zeta = \beta\sigma$ we find

$$z = w(\zeta) = \sum_{-\infty}^{\infty} a_n \beta^n \sigma^n = \sum_{-\infty}^{\infty} b_n(\beta)\, \sigma^n \qquad \ldots(49)$$

where $b_n(\beta) = a_n \beta^n$. Now we have the following relations

$$\left.\begin{aligned} \bar{w}w' &= \sum_{-\infty}^{\infty} c_n(\beta)\, \sigma^n,\ c_n = \sum_{-\infty}^{\infty} (n + r + 1)\, b_{n+r+1}(\beta)\, \bar{b}_r(\beta) \\ \bar{w}^2 w' &= \sum_{-\infty}^{\infty} d_n(\beta)\, \sigma^n,\ d_n = \sum_{-\infty}^{\infty} c_{n+r}(\beta)\, \bar{b}_r(\beta) \\ w\bar{w}w' &= \sum_{-\infty}^{\infty} e_n(\beta)\, \sigma^n,\ e_n = \sum_{-\infty}^{\infty} b_{-r}(\beta)\, c_{n+r}(\beta). \end{aligned}\right\} \qquad \ldots(50)$$

Assuming $\quad \psi(\zeta)\, w'(\zeta) = \sum_{-\infty}^{\infty} A_n \zeta^n \qquad \ldots(51)$

and using (49) to (51) in the boundary condition

$$\psi(\sigma)\, w'(\sigma) + \frac{1}{\sigma^2}\, \overline{\psi(\sigma)}\, \overline{w'(\sigma)}$$

$$= (p + q\bar{w} + rw)\, \bar{w}w' + \frac{1}{\sigma^2}\, (\bar{p} + \bar{q}w + \bar{r}\bar{w})\, w\bar{w}' \qquad \ldots(52)$$

we find

$$\sum_{-\infty}^{\infty} A_n \beta^n \sigma^n + \frac{\sum_{-\infty}^{\infty} \bar{A}_{-n-2}\sigma^n}{\beta^{n+2}} = \sum_{-\infty}^{\infty} [p c_n(\beta) + q d_n(\beta) + r e_n(\beta)$$

$$+ \bar{p}\bar{c}_{-n+2}(\beta) + \bar{q}\bar{d}_{-n+2}(\beta) + \bar{r}\bar{e}_{-n+2}(\beta)]\, \sigma^n. \qquad \ldots(53)$$

On the inner boundary we put $\beta = \beta_1$ and on the outer boundary $\beta = \beta_2$.

This system of equations uniquely determines the flexure function. Confocal elliptic and eccentric circular sections can be worked out using this method.

Simply Connected Region with an Internal Cut which is not a Physical Boundary

In this we use

$$z = \sum_{-\infty}^{\infty} a_n \zeta^n \qquad \ldots(54)$$

$\zeta = \beta\sigma$ which maps conformally a simply connected region with an internal cut which is not a physical boundary on the ring space $1 < |\zeta| \leqslant \beta$ in the ζ-plane. In this case we assume the flexure function in the form

$$\psi(\zeta) = \sum_{0}^{\infty} A_n(\zeta^n + 1/\zeta^n). \qquad \text{...(55)}$$

Using (54) and (55) in the boundary condition we get

$$E_n + \bar{E}_{-n-2} = (pc_n + qd_n + re_n) + (\bar{p}\bar{c}_{-n-2} + \bar{q}\bar{d}_{-n-2} + \bar{r}\bar{e}_{-n-2}) \qquad \text{...(56)}$$

where

$$E_n = (B_n + D_n),\ B_n = \sum_{-\infty}^{\infty} (-r + 1)\, A'_{n+r}\, b_{-r+1},$$

$$D_n = \sum_{-\infty}^{\infty} (n + r + 1)\, b_{n+r+1} A''_r,\ A'_n = A_n \beta^n,\ A''_n = A_n/\beta^n$$

the other constants are given by (50).

Cross-section an Ellipse

The function

$$z = w(\zeta) = a(\zeta + 1/\zeta) \qquad \text{...(57)}$$

$\zeta = \beta\sigma$ maps conformally the region interior to the ellipse onto the region $1 < |\zeta| \leqslant \beta$ in the ζ-plane. Using (57) in the boundary condition we find

$$\psi(\sigma)\, w' + \frac{1}{\sigma^2}\, \overline{\psi(\sigma)}\, \bar{w}' = a^3[q(2\beta - 1/\beta^3) + \bar{q}(\beta^3 - 2/\beta) + r(\beta^3 + 1/\beta) - \bar{r}/\beta^3]$$
$$+ (p - \bar{p})\, a^2\sigma + \frac{a^2}{\sigma}(\beta^2 - 1/\beta^2)(p + \bar{p}) + a^2(\bar{p} - p)/\sigma^3$$
$$+ \frac{a^3}{\sigma^2}[q(\beta^3 - 2/\beta) - r/\beta^3 + \bar{q}(2\beta - 1/\beta^3) + \bar{r}(\beta^3 + 1/\beta)]$$
$$+ a^3\sigma^2(q/\beta + r\beta - \bar{q}\beta - r/\beta)$$
$$+ \frac{a^3}{\sigma^4}(- q\beta - r/\beta + \bar{q}/\beta + \bar{r}\beta). \qquad \text{...(58)}$$

Assuming $\psi(\zeta) = A_0 + A_1(\zeta + 1/\zeta) + A_2(\zeta^2 + 1/\zeta^2)$ ...(59)

we get from (59) and (58)

$$\beta(A_0 - A_2) - (A_0 - \bar{A}_2)/\beta = a^2[q(2\beta - 1/\beta^3) + \bar{q}(\beta^3 - 2/\beta) r(\beta^3 + 1/\beta) - \bar{r}/\beta^3] \qquad \text{...(60)}$$

$$A_1\beta^2 - \bar{A}_1/\beta^2 = a(p - \bar{p}) \qquad \text{...(61)}$$

$$A_2\beta^3 - \bar{A}_2/\beta^3 = a^2(q/\beta + r\beta - \bar{q}\beta - \bar{r}/\beta). \qquad \text{...(62)}$$

Finally the complex flexure function for an elliptic section is given by

$$\Phi(z) = (A_0 - 2A_2) + A_1 z/a + A_2 z^2/a^2. \qquad \ldots(63)$$

With the usual notation the coordinates of shear centre are given by

$$I_x x_s + i I_y y_s = i \iint z\phi \, dx \, dy. \qquad \ldots(64)$$

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