ON BOUNDS OF 'USEFUL' INFORMATION MEASURES*

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For a probability distribution $\mathcal{P} = (p_1, p_2, \ldots, p_N)$, $p_i > 0$, $\sum_{i=1}^{N} p_i = 1$; and a utility distribution $\mathcal{U} = (u_1, u_2, \ldots, u_N)$, where u_i (> 0) is the utility of the event with probability p_i, some bounds involving Belis-Guiaşu and generalized 'useful' information meaures for $\mathcal{P}$ and its power distribution $\mathcal{P}^\beta$ are obtained.

INTRODUCTION

Consider a probability distribution $\mathcal{P} = (p_1, p_2, \ldots, p_N)$, $p_i > 0$, $\sum_{i=1}^{N} p_i = 1$; and a utility distribution $\mathcal{U} = (u_1, u_2, \ldots, u_N)$, where u_i (> 0) is the utility of the event with probability p_i for $i = 1, 2, \ldots, N$. The Belis-Guiaşu (1968) quantitative-qualitative measure (called 'useful' information by Longo 1972) is given by

$$H(\mathcal{U}; \mathcal{P}) = - \sum_{i=1}^{N} u_i p_i \log p_i. \qquad \ldots(1)$$

In a recent work, Sharma *et al.* (1978a, b) characterized the generalized measure of 'useful' information

$$H^\alpha(\mathcal{U}; \mathcal{P}) = (2^{1-\alpha} - 1)^{-1} \left\{ \sum_{i=1}^{N} u_i p_i (p_i^{\alpha-1} - 1) \right\} \qquad \ldots(2)$$

where $\alpha \neq 1$. It may be observed that if the utilities are ignored, say by setting each $u_i = 1$, then (1) becomes the Shannon's entropy (Shannon and Weaver 1949)

$$H(\mathcal{P}) = - \sum_{i=1}^{N} p_i \log p_i \qquad \ldots(3)$$

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and (2) reduces to the Havrda-Charvat (1967) entropy of type α,

$$H^{\alpha}(\mathcal{P}) = (2^{1-\alpha} - 1)^{-1} \left\{ \sum_{i=1}^{N} p_i(p_i^{\alpha-1} - 1) \right\}, \quad \text{for } \alpha > 0. \qquad \ldots(4)$$

Recently, Mitter and Mathur (1972) established some comparison theorems involving $H(\mathcal{P})$ and $H(\mathcal{P}^{\beta})$, where

$$H(\mathcal{P}^{\beta}) = \frac{-\sum_{i=1}^{N} p_i^{\beta} \log_2 p_i^{\beta}}{\sum_{i=1}^{N} p_i^{\beta}}, \qquad \ldots(5)$$

$\mathcal{P}^{\beta} = (p_1^{\beta}, p_2^{\beta}, \ldots, p_N^{\beta})$, $p_i^{\beta} > 0$ for $\beta > 0$, being the power distribution of $\mathcal{P}$.

In this communication, some bounds on 'useful' information measures (1) and (2) for $\mathcal{P}$ and $\mathcal{P}^{\beta}$ are obtained.

BOUNDS FOR $H(\mathcal{U}; \mathcal{P})$

Theorem 1 — With the notations given above

$$0 \leqslant H(\mathcal{U}; \mathcal{P}) \leqslant \bar{u} \log \frac{U}{\bar{u}} \qquad \ldots(6)$$

where

$$\bar{u} = \sum_{i=1}^{N} u_i p_i \text{ and } U = \sum_{i=1}^{N} u_i. \qquad \ldots(7)$$

PROOF : We have the inequality (refer Beckenbach and Bellman 1971, p. 17)

$$\log \frac{\sum_{i=1}^{N} a_i b_i}{\sum_{i=1}^{N} b_i} \leqslant \frac{\sum_{i=1}^{N} a_i b_i \log a_i}{\sum_{i=1}^{N} a_i b_i}$$

for $a_i > 0$ and $b_i > 0$; with equality iff all a_i's are equal.

Setting $b_i = u_i$ and $a_i = p_i$, $u_i > 0$, $p_i > 0$, $i = 1, 2, \ldots, N$, we get

$$H(\mathcal{U}; \mathcal{P}) \leqslant \bar{u} \log \frac{U}{\bar{u}}. \blacksquare$$

Corollary — Max. $H(\mathcal{U}; \mathcal{P}) = \frac{U}{N} \log N$. ...(8)

The result is a direct consequence of right-hand equality in (6) when $p_i = \frac{1}{N}$ for all i. ∎

Remark : When all $u_i = 1$, then (6) reduces to

$$0 \leqslant H(\mathscr{P}) \leqslant \log N$$

which is well known for Shannon's measure.

Comparison Theorems

Theorem 2 — With usual notations

$$H(\mathscr{U}; \mathscr{P}) \begin{cases} > C_\alpha H^\alpha(\mathscr{U}; \mathscr{P}) & \text{for } \alpha > 1 \\ < C_\alpha H^\alpha(\mathscr{U}; \mathscr{P}) & \text{for } \alpha < 1 \end{cases} \quad \ldots(9)$$

where

$$C_\alpha = (2^{1-\alpha} - 1)(1 - \alpha)^{-1}, \ \alpha \neq 1.$$

Proof : From the inequality (refer Hardy *et al.* 1934, p. 117)

$$\log x_i \begin{cases} < \dfrac{x_i^r - 1}{r} & \text{for } r > 0 \\ > \dfrac{x_i^r - 1}{r} & \text{for } r < 0 \end{cases} \quad \ldots(10)$$

where $x_i > 0$ and $x_i \neq 1$ for all i.

Setting $x_i = p_i \ (> 0)$, $r = \alpha - 1 \ (\alpha \neq 1)$ and considering the case $\alpha > 1$, (10) gives

$$\log p_i < \frac{p_i^{\alpha-1} - 1}{\alpha - 1}. \quad \ldots(11)$$

Multiplying (11) by $u_i p_i$ ($u_i > 0$) and summing over all i, we get

$$\sum_{i=1}^{N} u_i p_i \log p_i < \frac{\sum_{i=1}^{N} u_i p_i (p_i^{\alpha-1} - 1)}{\alpha - 1}$$

or

$$-\frac{H(\mathscr{U}; \mathscr{P})}{2^{1-\alpha} - 1} > \frac{H^\alpha(\mathscr{U}; \mathscr{P})}{\alpha - 1}$$

since $2^{1-\alpha} - 1 < 0$ for $\alpha > 1$.

This establishes (9) for the case $\alpha > 1$.

Similar substitutions in (10) for $r < 0$, prove (9) for the case $\alpha < 1$. ∎

Corollary — When all $u_i = 1$, then the above comparison theorem reduces to the following results involving the Shannon's entropy and the Havrda-Charvat entropy of type α :

$$H(\mathcal{P}) \begin{cases} > C_\alpha H^\alpha(\mathcal{P}) & \text{for } \alpha > 1 \\ < C_\alpha H^\alpha(\mathcal{P}) & \text{for } \alpha < 1 \end{cases} \qquad \text{...(12)}$$

where

$$C_\alpha = (2^{1-\alpha} - 1)(1 - \alpha)^{-1}, \; \alpha \neq 1.$$

Comparison of Results for Power Distributions

Theorem 3 — For the power distribution

$\mathcal{P}^\beta = (p_1^\beta, p_2^\beta, ..., p_N^\beta)$, $p_i^\beta > 0$, $(\beta > 0)$, the following holds :

$$H(\mathcal{U}; \mathcal{P}) \begin{cases} < \beta^{-1} H(\mathcal{U}; \mathcal{P}^\beta) & \text{for } \beta < 1 \\ = \beta^{-1} H(\mathcal{U}; \mathcal{P}^\beta) & \text{for } \beta = 1 \\ > \beta^{-1} H(\mathcal{U}; \mathcal{P}^\beta) & \text{for } \beta > 1. \end{cases} \qquad \text{...(13)}$$

Proof : We have (refer Hardy *et al.* 1934, p. 78)

$$\frac{\sum_{i=1}^{N} p_i \log a_i}{\sum_{i=1}^{N} p_i} \leqslant \frac{\sum_{i=1}^{N} p_i a_i \log a_i}{\sum_{i=1}^{N} p_i a_i}, \quad a_i > 0 \qquad \text{...(14)}$$

with equality iff a_i's are equal.

Setting $a_i = p_i^{\beta-1}$ in (14), where $\beta \neq 1$ is a real number such that all $p_i^{\beta-1}$ are not equal, (13) is easily established. ∎

Writing $\mathcal{P}^\gamma$ as

$$\mathcal{P}^\gamma = \{(p_1^\beta)^{\gamma/\beta}, (p_2^\beta)^{\gamma/\beta}, ..., (p_n^\beta)^{\gamma/\beta}\}$$

it follows as an immediate consequence of the above theorem that if we replace $\mathcal{P}$, β, $\mathcal{P}^\beta$ in (13) by $\mathcal{P}^\beta$, $\frac{\gamma}{\beta}$, $\mathcal{P}^\gamma$ respectively, then

$$\beta^{-1} H(\mathcal{U}; \mathcal{P}^\beta) \geqslant \gamma^{-1} H(\mathcal{U}; \mathcal{P}^\beta) \;\; \text{for } \beta \leqslant \gamma \; (\beta, \gamma > 0).$$

Further if $u_i = 1$ for each i, then the results in (13) and (15) reduce to the corresponding comparison theorems (Mitter and Mathur 1972) for the Shannon's entropy.

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