ON A MIXED BOUNDARY VALUE PROBLEM IN ELASTICITY

by G. K. DHAWAN, *Department of Mathematics, M. A. College of Technology, Bhopal* 462007

and

I. K. GOUTAM, *Department of Mathematics, S. V. Govt. Polytechnic, Bhopal* 462002

(*Received* 3 *March* 1978)

Stresses and displacements are found in the plane of a small line stress-free crack which lies at the interface between a strip and a foundation. The strip is assumed to be rigidly bonded to the foundation, except in the interior of the line crack, and top edge of the strip is deformed by a known pressure loading. Solution of the elastic field equations by Fourier transforms leads to four coupled integral equations which, by means of a general technique of Noble, are reduced to two coupled Fredholm integral equations. Expressions for various quantities of physical interest are derived for the case where the thickness of the strip is much greater than the crack radius by finding the iterative solutions of these equations. For the values of this thickness near unity, the integral equations have been solved numerically.

1. INTRODUCTION

A wide class of problems in mechanics require the analysis of stress in the neighbourhood of localized imperfection. The motivation of such studies lies in their application to fracture. The localized imperfections such as cracks or hard inclusions form the nucleus of the fracture initiation and propagation in the medium. The interface in bonded material always contains some imperfections and, if the bond is not sufficiently strong, generally the mode of fracture is delamination along the interface initiating at these imperfections. In analyzing such problems, it may be sufficient to study the perturbed stress state caused by the imperfection (generally a crack) located on the interface.

Lowengrub (1966) considered the problem of finding the stress distribution in the plane of a line crack symmetrically located between the edge of a strip. Extending methods employed by Sneddon (1946) he reduces the problem to finding the solution of dual integral equations. By means of a well-known technique due to Lebdev and Uflyand (1958) the dual integral equations are then reduced to a standard Fredholm integral equation using Noble technique (1963) for the case of a strip. The stresses and displacements are then evaluated.

The key to Lowengrub's problem is the symmetry principle of Sneddon. If the crack is not centrally located, the resulting analysis, gives four integral equations in two unknowns. In this case, the technique of Lebdev and Uflyand cannot be used.

We consider such an asymmetrical problem in this paper. Specifically we consider a thick strip of elastic incompressible material to be bonded molecularly to a foundation. The bond is assumed to be imperfect to the extent that a small line crack exists between strip and foundation. In the bond line the portion interior to the crack is stress free while that exterior to the crack is rigidly attached to the foundation. The other edge of the strip is deformed by means of a known pressure distribution.

The solution of the field equations by means of Fourier Transforms leads to four coupled integral equations in two unknown which are further reduced to two coupled Fredholm equations. When the thickness of the strip is large with respect to crack, iterative solutions of these equations are obtained to yield expressions for quantities of physical interest. The values of these quantities for the values of thickness nearly equal to unity are derived from numerical solutions of the basic Fredholm integral equations. The results are illustrated graphically (Figs. 2-7).

2. The Boundary Value Problem

Formulated in a Cartesian coordinates system (x, y), the strip occupies the region $0 \leqslant x \leqslant \infty$, $0 \leqslant y \leqslant h$. The strip is attached to the foundation at $y = 0$ and the line crack is specified by $y = 0$ and $0 \leqslant x \leqslant 1$. The plane $y = h$ is stress free except for a known pressure distribution of the form $P_0 H(a - x)$, $0 < a < \infty$, where $H(x)$ is Heavside step function.

The problem is to determine the stress and displacement fields which satisfy the elastic field equations

$$2(1-\eta)\frac{\partial^2 u}{\partial x^2} + (1-2\eta)\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 0 \qquad \text{...(2.1)}$$

$$(1-2\eta)\frac{\partial^2 v}{\partial x^2} + 2(1-\eta)\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} = 0 \qquad \text{...(2.2)}$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \text{...(2.3)}$$

$$\sigma_y = \frac{2\mu}{1-2\eta}\left\{(1-\eta)\frac{\partial v}{\partial y} + \eta\frac{\partial u}{\partial x}\right\} \qquad \text{...(2.4)}$$

where $\mu = E/2(1-\eta) : \eta$ is Poisson's ratio and E, Young's modulus of the elastic material.

To complete the problem, we write the boundary conditions, which are

$$\sigma_y(x, h) = P_0 H(a - x) \qquad \text{...(2.5)}$$

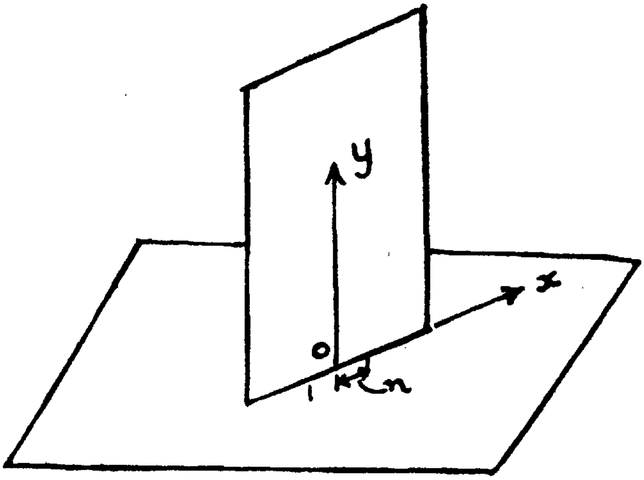

FIG. 1.

$$\tau_{xy}(x, h) = 0 \quad \ldots(2.6)$$

$$\sigma_y(x, 0) = \tau_{xy}(x, 0) = 0, \quad 0 \leqslant x < 1 \quad \ldots(2.7)$$

$$u(x, 0) = v(x, 0) = 0, \quad 1 < x < \infty. \quad \ldots(2.8)$$

The relation (2.7) states that the interior of the crack is stress free while (2.8) specify that the exterior is rigidly attached to the foundation.

3. Solution of the Problem

Because of the simplicity of these equations and because we use standard Fourier transform technique to solve equilibrium equations, we merely write down solutions that satisfy (2.1) to (2.6). The displacements and stresses of direct concern are

$$2u(x, y) = \int_0^\infty \sin(\xi x)\Big[B(\xi)\{f_1(\xi, y)e^{\xi y} + e^{-\xi y+2\xi h}\} - D(\xi)\{e^{\xi y-2\xi h} + f_2(\xi, y)e^{-\xi y}\} - \frac{3P_0 \sin(\xi a)}{\xi^2 E}\cosh\xi(h-y)\Big]d\xi \quad \ldots(3.1)$$

$$2v(x, y) = \int_0^\infty \cos(\xi x)\Big[B(\xi)\{f_2(\xi, y)e^{\xi y} + e^{-\xi y+2\xi h}\} + D(\xi)\{e^{\xi y-2\xi h} + f_1(\xi, y)e^{-\xi y}\} - \frac{3P_0 \sin(\xi a)}{\xi^2 E}\sinh\xi(h-y)\Big]d\xi \quad \ldots(3.2)$$

$$\frac{2\tau_{xy}(x,y)}{E} = \int_0^\infty \xi \sin(\xi x)\Big[B(\xi)\{f_1(\xi,y)\,e^{\xi y} - e^{-\xi y+2\xi h}\}$$

$$+ D(\xi)\{-e^{\xi y-2\xi h} + f_2(\xi,y)\,e^{-\xi y}\} + \frac{3P_0 \sin(\xi a)}{\xi^2 E}\sinh \xi(h-y)\Big]\,d\xi \qquad \ldots(3.3)$$

$$\frac{2\sigma_y(x,y)}{E} = \int_0^\infty \xi \cos(\xi x)\Big[B(\xi)\{f_2(\xi,y)\,e^{\xi y} - e^{-\xi y+2\xi h}\}$$

$$+ D(\xi)\{e^{\xi y-2\xi h} - f_1(\xi,y)\,e^{-\xi y}\} + \frac{3P_0 \sin(\xi a)}{\xi^2 E}\cosh \xi(h-y)\Big]\,d\xi \qquad \ldots(3.4)$$

where

$$f_1(\xi,y) = 1 - 2\xi(h-y)$$

$$f_2(\xi,y) = 1 + 2\xi(h-y).$$

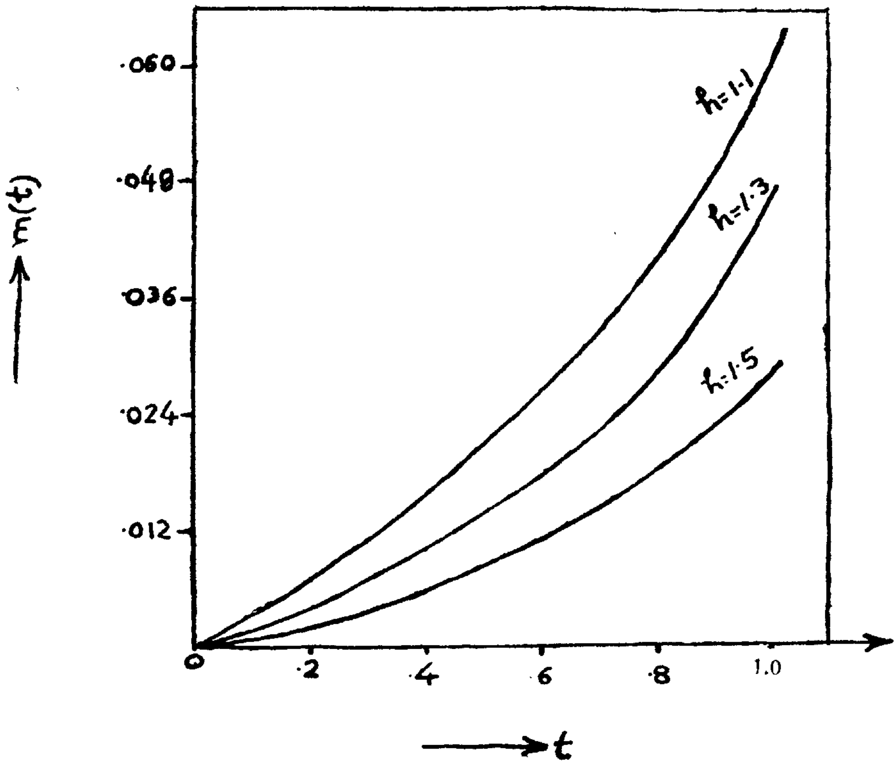

FIG. 2. The variation of the function $m(t)$ with t and h.

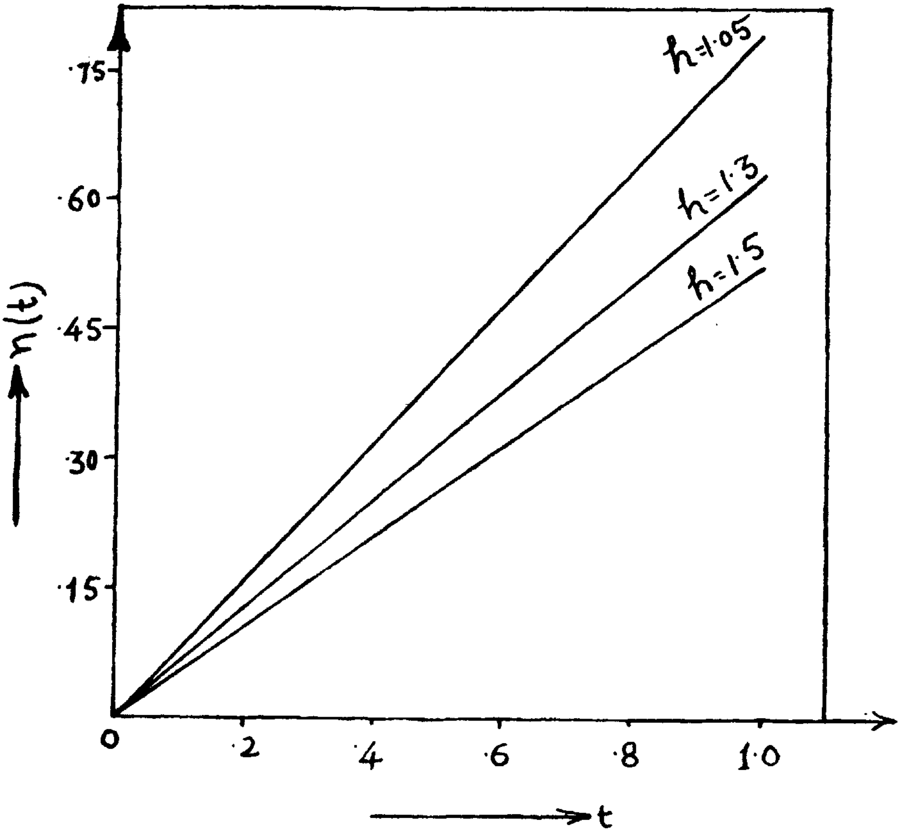

FIG. 3. The variation of the function $n(t)$ with t and h.

The two constants B and D are to be determined by conditions (2.7) and (2.8). Now evaluating (3.1) – (3.4) at $y = 0$ and taking the integrals of (3.1) and (3.2) i.e. the part of integrand which multiplies sin (ξx) in (3.1) and cos (ξx) in (3.2) equal to the quantities $2M(\xi)$ and $2N(\xi)$ respectively. Solving the system of equations for B and D, substituting in (3.3) and (3.4), which are then evaluated at $y = 0$ lead to

$$u(x, 0) = \int_0^\infty M(\xi) \,.\, \sin(\xi x)\, d\xi \qquad ...(3.5)$$

$$v(x, 0) = \int_0^\infty N(\xi) \cos(\xi x)\, d\xi \qquad ...(3.6)$$

$$-\frac{3\tau_{xy}(x, 0)}{2E} = \int_0^\infty \frac{\xi \sin \xi x}{\xi^2 h^2 + \cosh^2 \xi h} \Big[M(\xi)\{\xi h + \tfrac{1}{2} \sinh 2\xi h\} + \xi^2 h^2 N(\xi) + \frac{3P_0 \sin(\xi a)}{2\xi^2 E} \,.\, \xi h \cosh \xi h \Big] d\xi \qquad ...(3.7)$$

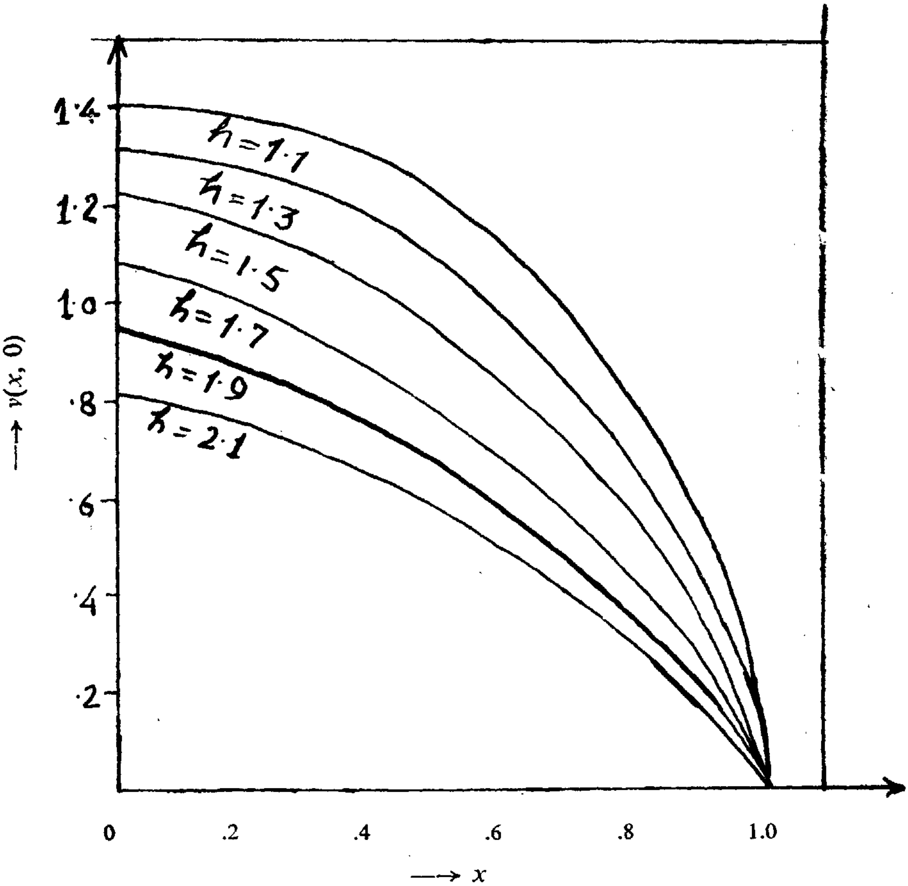

FIG. 4. The variation of the normal displacement with t and h.

$$-\frac{3\sigma v(x,0)}{2E} = \int_0^\infty \frac{\xi\cos\xi x}{\xi^2h^2+\cosh^2\xi h}\Big[\xi^2h^2M(\xi)+N(\xi)\{\tfrac{1}{2}\sinh 2\xi h-\xi b\}$$

$$-\frac{3P_0\sin(\xi a)}{2\xi^2E}\{\cosh\xi h+\xi h\sinh\xi h\}\Big]d\xi. \qquad \text{...(3.8)}$$

Now substituting (3.5) – (3.8) into the boundary conditions (2.7) and (2.8) lead to integral equations

$$\int_0^\infty M(\xi)\sin(\xi x)\,d\xi = 0, \qquad 1 < x < \infty \qquad \text{...(3.9)}$$

$$\int_0^\infty N(\xi)\cos(\xi x)\,d\xi = 0, \qquad 1 < x < \infty \qquad \text{...(3.10)}$$

$$\int_0^\infty [M(\xi)\{1 + H_1(\xi h)\} + H_2(\xi h) N(\xi) + R(\xi, a)\,\Omega_1(\xi h)] \,.\, \xi \sin(\xi x)\, d\xi = 0,$$

$$0 \leqslant x < 1 \qquad \ldots(3.11)$$

$$\int_0^\infty [M(\xi) H_2(\xi h) + N(\xi)\{1 + H_3(\xi h)\} - R(\xi, a)\,\Omega_2(\xi h)] \,.\, \xi \cos(\xi x)\, d\xi = 0,$$

$$0 \leqslant x < 1 \qquad \ldots(3.12)$$

with

$$1 + H_1(\xi h) = \frac{\xi h + \frac{1}{2}\sinh 2\xi h}{(\xi h)^2 + \cosh^2 \xi h}$$

$$1 + H_3(\xi h) = \frac{\frac{1}{2}\sinh 2\xi h - \xi h}{(\xi h)^2 + \cosh^2 \xi h}$$

$$H_2(\xi h) = \frac{(\xi h)^2}{(\xi h)^2 + \cosh^2 \xi h}$$

$$R(\xi, a) = \frac{3P_0 \sin(\xi a)}{2\xi^2 E}.$$

4. Reduction to Fredholm Integral Equations

We have to find the solutions of the integral eqns. (3.9) – (3.12). We shall presently show that these equations can be reduced to simultaneous Fredholm integral equations of the second kind which are best solved by numerical methods. However, in the case when $h \geqslant 1$ and the integrals

$$\int_0^\infty u^n H_j(u)\, du,\ n = 0, 1, 2, \ldots;\ j = 1, 2, 3$$

are convergent, interative solution can be derived.

Let the trial solutions be

$$M(\xi) = \int_0^1 m(t)\, J_1(\xi t)\, dt = \frac{-m(1)}{\xi} J_0(\xi t) + \frac{1}{\xi}\int_0^1 m'(t) J_0(\xi t)\, dt,\ m(0) = 0 \qquad \ldots(4.1)$$

$$N(\xi) = \int_0^1 n(t)\, J_0(\xi t)\, dt. \qquad \ldots(4.2)$$

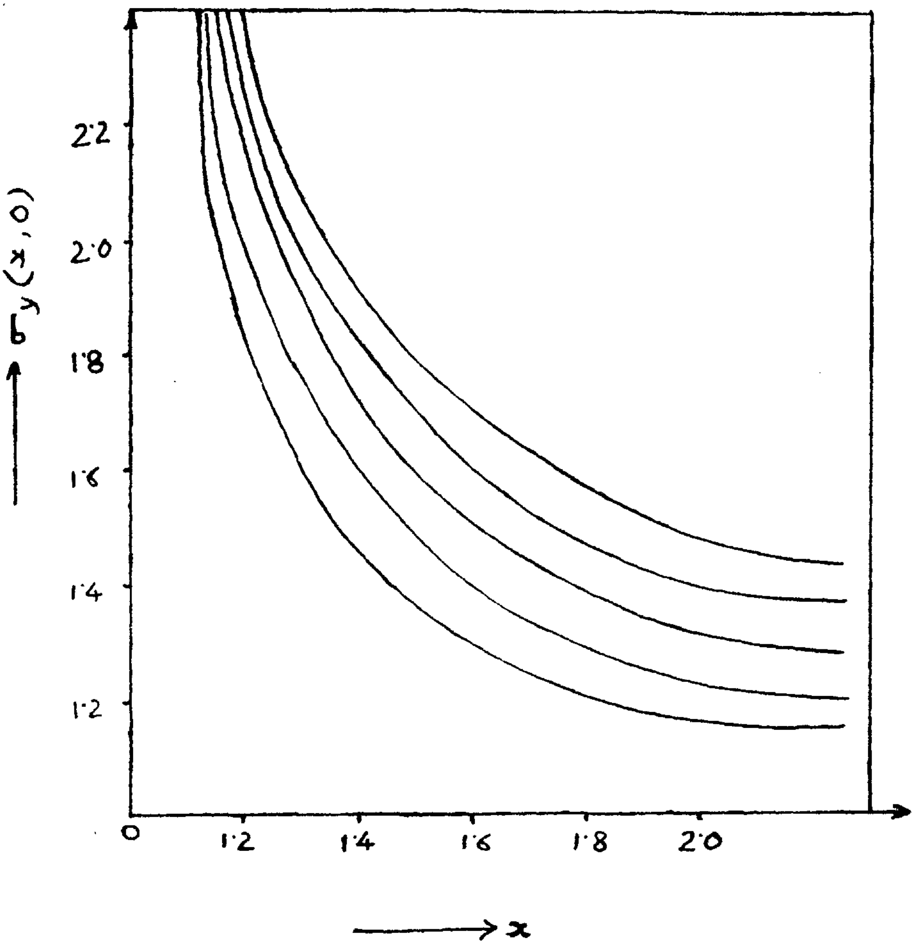

FIG. 5. The variation of normal stress with x and h.

It is easy to see that (3.9) and (3.10) are satisfied for the choice of $M(\xi)$ and $N(\xi)$. We can write (3.11) and (3.12) as

$$\left.\begin{aligned}\frac{d}{dx}\int_0^\infty N(\xi)\sin\xi x\,d\xi = \psi(x) &= -\int_0^\infty M(\xi)\,[H_1(\xi h) + H_2(\xi h)\,N(\xi) \\ &\qquad + R(\xi, a)\,\Omega_1(\xi h)]\,\xi\sin(\xi x)\,d\xi \\ \frac{d}{dx}\int_0^\infty M(\xi)\sin\xi x\,d\xi = \phi(x) &= -\int_0^\infty M(\xi)\,[H_2(\xi h) + H_3(\xi h) \\ &\qquad - R(\xi, a)\,\Omega_2(\xi h)]\,\xi\cos(\xi x)\,d\xi.\end{aligned}\right\}\quad \dots(4.3)$$

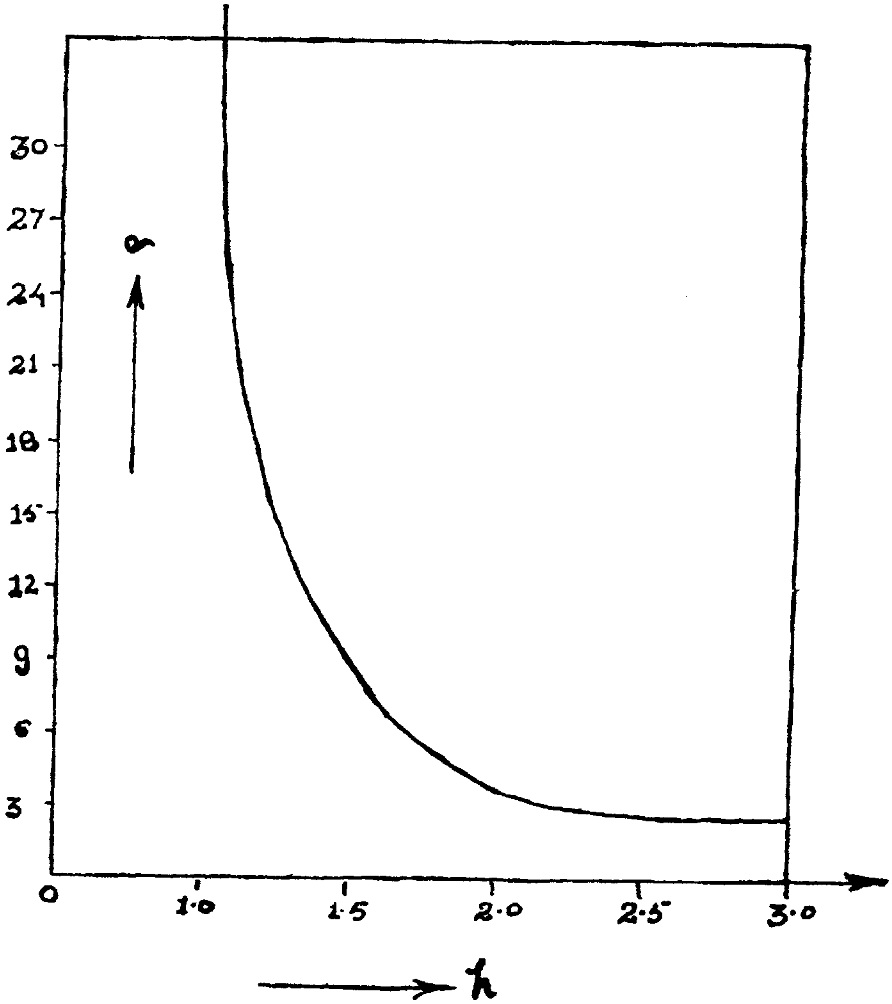

FIG. 6. The variation of normal intensity factor with h.

Substituting the values of $M(\xi)$ and $N(\xi)$ from (4.1) and (4.2), changing the order of integration, we get

$$\frac{d}{dx}\int_0^x \frac{n(t)}{\sqrt{x^2 - t^2}} = \psi(x) \qquad \ldots(4.4)$$

$$\int_0^x \frac{m'(t)}{\sqrt{x^2 - t^2}} = \phi(x). \qquad \ldots(4.5)$$

The solutions of these equations are

$$n(t) = \frac{2t}{\pi} \int_0^t \frac{\psi(x)}{\sqrt{t^2 - x^2}} dx \qquad \ldots(4.6)$$

$$m'(t) = \frac{2}{\pi} \cdot \frac{d}{dt} \int_0^t \frac{x\phi(x)}{\sqrt{t^2 - x^2}} dx. \qquad \ldots(4.7)$$

Integrating (4.6), we have

$$m(t) = \frac{2}{\pi} \int_0^t \frac{x\phi(x)}{\sqrt{t^2 - x^2}} dx. \qquad \ldots(4.8)$$

Now putting the values of $M(\xi)$ and $N(\xi)$ and applying the inversion theorem (Tricomi 1957) we get the following Fredholm integral equations :

$$m(t) + \int_0^1 m(u) L_1(t, u) du + \int_0^1 n(u) L_2(t, u) du$$

$$= - t \int_0^\infty \xi R(\xi, a) \Omega_1(\xi h) J_1(\xi t) d\xi \qquad \ldots(4.9)$$

$$n(t) + \int_0^1 m(u) L_2(t, u) du + \int_0^1 n(u) L_4(t, u) du$$

$$= t \int_0^\infty \xi R(\xi, a) \Omega_2(\xi h) J_0(\xi t) d\xi \qquad \ldots(4.10)$$

where

$$L_1(t, u) = t \int_0^\infty \xi H_1(\xi h) J_1(\xi t) J_1(\xi u) d\xi$$

$$L_2(t, u) = t \int_0^\infty \xi H_2(\xi h) J_1(\xi t) J_0(\xi u) d\xi$$

$$L_3(t, u) = t \int_0^\infty \xi H_2(\xi h) J_0(\xi t) J_1(\xi u) d\xi$$

$$L_4(t, u) = t \int_0^\infty \xi H_3(\xi h) J_0(\xi t) J_0(\xi u) d\xi.$$

These determine the unknown functions $m(t)$ and $n(t)$.

5. Solution for Large h

When $h \gg 1$ we can write the expansions of Bessel functions in power series, taking $z = \xi h$,

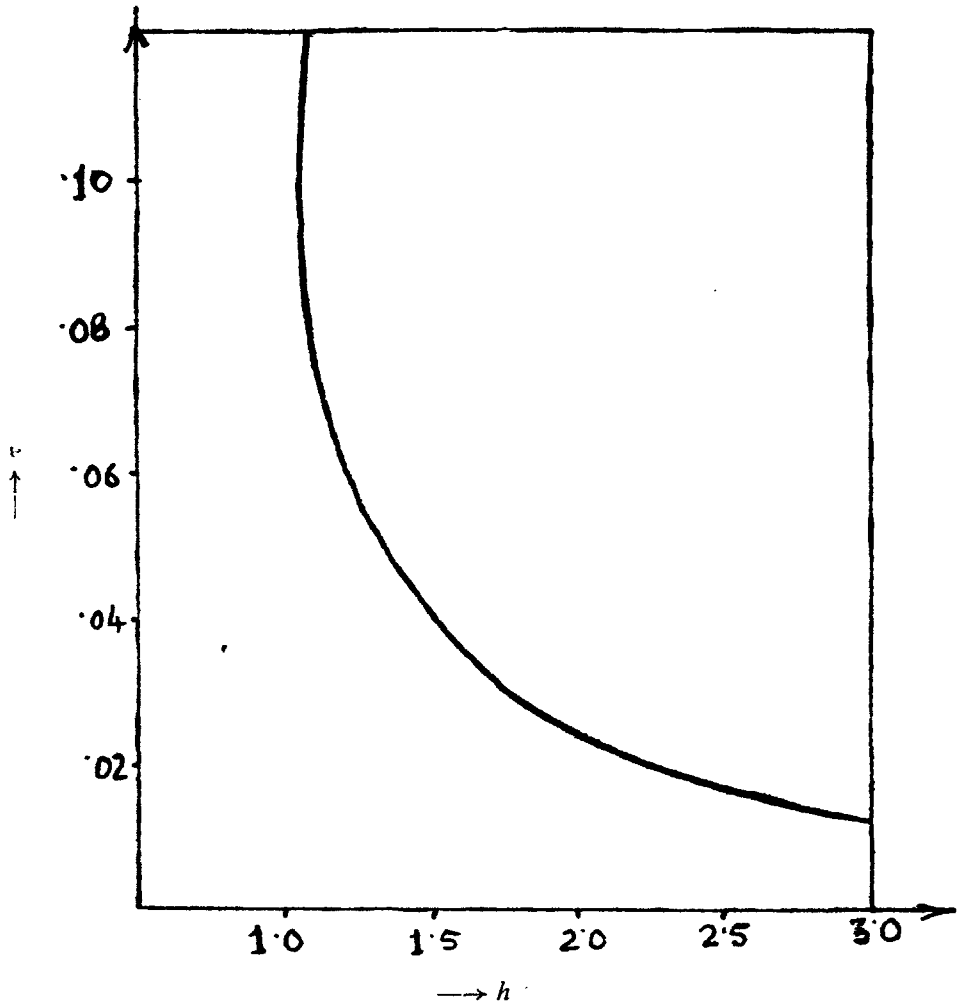

FIG. 7. The variation of shearing intensity factor with h.

$$L_1(t, u) = \frac{A_0 u t^2}{h^4} + \frac{A_1 u t^2}{h^6}(u^2 + t^2) + o(h^{-8}) \qquad \text{...(5.1)}$$

$$L_2(t, u) = \frac{B_0 u t}{h^3} + \frac{B_1 u t}{h^5}(u^2 + 2t^2) + o(h^{-7}) \qquad \text{...(5.2)}$$

$$L_3(t, u) = \frac{C_0 t^2}{h^3} + \frac{A_1 t^2}{h^5}(2u^2 + t^2) + o(h^{-7}) \qquad \text{...(5.3)}$$

$$L_4(t, u) = \frac{D_0 t}{h^2} + \frac{D_1 t}{h^4}(u^2 + t^2) + o(h^{-6}) \qquad \text{...(5.4)}$$

where

$$A_0 = \tfrac{1}{4} \int_0^\infty z^3 H_1(z)\, dz = -\,1.401209$$

$$A_1 = -\,1/32 \int_0^\infty z^5 H_1(z)\, dz = 1.801732$$

$$B_0 = C_0 = \tfrac{1}{2} \int_0^\infty z^2 H_2(z)\, dz = 1.201132$$

$$B_1 = C_1 = -\,1/16 \int_0^\infty z^4 H_2(z)\, dz = -\,1.270245$$

$$D_0 = \int_0^\infty z H_3(z)\, dz = -\,2.969544$$

$$D_1 = -\,\tfrac{1}{4} \int_0^\infty z^3 H_3(z)\, dz = 2.692148.$$

The right-hand sides in (4.9) and (4.10), when expanded, yield

$$t \int_0^\infty \xi R(\xi, a)\, \Omega_1(\xi h)\, J_1(\xi t)\, d\xi = \frac{3P_0 t}{2E} \left[\frac{t}{2h} \mathbf{\Lambda}_0(\rho) - \frac{t^3}{16h^3} \mathbf{\Lambda}_1(\rho) + \frac{t^5}{384h^5} \mathbf{\Lambda}_2(\rho) + o(h^{-7}) \right] \quad \ldots(5.5)$$

$$t \int_0^\infty \xi R(\xi, a)\, \Omega_2(\xi h)\, J_0(\xi t)\, d\xi = \frac{3P_0 t}{2E} \left[\phi_0(\rho) - \frac{t^2}{4h^2} \phi_1(\rho) + \frac{t^4}{64h^4} \phi_2(\rho) + o(h^{-6}) \right] \quad \ldots(5.6)$$

where

$$\rho = \frac{a}{h}$$

and

$$\left. \begin{aligned} \mathbf{\Lambda}_k(\rho) &= \int_0^\infty \lambda^{2k} \sin(\lambda\rho)\, \Omega_1(\lambda)\, d\lambda \\ \phi_k(\rho) &= \int_0^\infty \lambda^{2k-1} \sin(\lambda\rho)\, \Omega_2(\lambda)\, d\lambda. \end{aligned} \right\} \quad \ldots(5.7)$$

A standard perturbation solution of (4.9) and (4.10) in terms of $h^{-\frac{1}{2}}$, using the above (5.1) – (5.7) is :

$$m(t) = \frac{-3P_0}{4E}\Big[\frac{t^2\Lambda_0(\rho)}{h} - \frac{1}{h^3}\{C_0t^2\phi_0(\rho) - \tfrac{1}{8}t^4\Lambda_1(\rho)\} + \frac{1}{h^5}\{-\tfrac{1}{4}A_0t^2\Lambda_0(\rho)$$
$$+ C_1t^2(1+t^2)\,\phi_0(\rho)$$
$$+ \frac{1}{192}t^6\,\Lambda_2(\rho) - C_0t^2(\tfrac{1}{2}D_0\phi_0(\rho) + \tfrac{1}{8}\phi_1(\rho)\Big] + o(h^{-7}) \qquad ...(5.8)$$

$$n(t) = \frac{3P_0}{4E}\Big[t\phi_0(\rho) - \frac{1}{h^2}\{\tfrac{1}{2}D_0t\phi_0(\rho) + \tfrac{1}{4}t^3\phi_1(\rho)\} + \frac{1}{h^4}\Big\{\tfrac{1}{8}B_0t\Lambda_0(\rho)$$
$$-\frac{D_0}{4}t(1+2t^2)\,\phi_0(\rho) + \frac{1}{64}t^5\phi_2(\rho) + D_0(t)(\tfrac{1}{4}D_0\phi_0(\rho) + \frac{1}{16}\phi_1(\rho)\Big\}\Big]$$
$$+ o(h^{-6}). \qquad ...(5.9)$$

6. Boundary Values of the Stresses and Displacements

(1) *Displacements*

Substituting (4.1) and (4.2) into (3.9) and (3.10) respectively, we wet

$$u(x, 0) = x\int_0^1 \frac{t^{-1}m(t)}{\sqrt{t^2 - x^2}}\,dt \qquad ...(6.1)$$

$$v(x, 0) = \int_x^1 \frac{n(t)}{\sqrt{t^2 - x^2}}\,dt. \qquad ...(6.2)$$

Putting the values of $m(t)$ and $n(t)$ and evaluating the integral we get

$$u(x, 0) = \frac{-3P_0x}{4E}\sqrt{1-x^2}\Big[\alpha_1 + \alpha_2(1+2x^2) + \alpha_3(4+2x^2)$$
$$+ \alpha_4\frac{8x^4 + 4x^3 + 3}{2880x^5} + o(h^{-7})\Big] \qquad ...(6.3)$$

$$v(x, 0) = \frac{3P_0}{2E}\sqrt{1-x^2}\,[\beta_1 + \beta_2(1+2x^2) + \beta_3(2+4x^2) + o(h^{-6})] \qquad ...(6.4)$$

where

$$\alpha_1 = \frac{\Lambda_0(\rho)}{h} - \frac{C_0\phi_2(\rho)}{h^3} + \frac{1}{h^5}\Big\{-\tfrac{1}{4}A_0\Lambda_0(\rho) - C_0\frac{(4D_0\phi_0 + \phi_1)}{8} + \frac{1}{192}\Big\};$$

$$\alpha_2 = \frac{\Lambda_1(\rho)}{24h^3};\quad \alpha_3 = \frac{C_1\phi_0(\rho)}{h^5};\quad \alpha_4 = \frac{\Lambda_1(\rho)}{h^5};$$

$$\beta_1 = \phi_0(\rho) - \frac{1}{2h^2} D_0\phi_0(\rho) + \frac{1}{h^4} \left\{\frac{D_1B_0\Lambda_0(\rho)}{8} - \tfrac{1}{4} D_0\phi_0(\rho) + \frac{D_0^2}{4}\phi_0(\rho) + \frac{D_0\phi_1(\rho)}{16} + \frac{\phi_2(\rho)}{64}\right\}$$

$$\beta_2 = -\frac{1}{2h^2}\phi_1(\rho); \quad \beta_3 = -\frac{D_0}{12h^4}\phi_0(\rho).$$

(2) *Stresses*

The stresses are found in a similar but slightly more complicated manner. We avoid details and only give the final results for the shear stress and normal stress. They are

$$\sigma_y(x, 0)]_{x>1} = P_0 \left[\left(\frac{x}{\sqrt{x^2-1}}\right)(\gamma_1 + \gamma_2(1+2x^2) + \gamma_3 + \gamma_4 x^2 + \gamma_5 \frac{x(3x^2-1)}{\sqrt{x^2-1}}\right] + o(h^{-6}) \quad \ldots(6.5)$$

$$\tau_{xy}(x, 0)]_{x>1} = \tfrac{1}{2} P_0 \left[\left(\frac{2x^2-1}{\sqrt{x^2-1}}\right)\delta_1 + \delta_2 x + \delta_3 \frac{x^2(2x^2-1)}{\sqrt{x^2-1}} + \delta_4 x^3 + \delta_5(4x^4 + x^2 - 2) + o(h^{-6})\right] \quad \ldots(6.6)$$

where

$$\gamma_1 = \phi_0 - \frac{D_0\phi_0}{2h^2} + \frac{1}{h^4}\left(\frac{B_0}{8}\Lambda_0 - \frac{D_0}{4}\phi_0 + \tfrac{1}{4} D_0^2\phi_0 + \frac{1}{16} D_0\phi_1\right);$$

$$\gamma_2 = -\frac{1}{9h^2}\phi_1;$$

$$\gamma_3 = \frac{D_0\phi_0}{2h^2} - \frac{D_0^2\phi_0}{4h^4} - \frac{D_0\phi_1}{12h^4} - \frac{D_1\phi_0}{4h^4} + \frac{C_0\Lambda_0}{8h^4};$$

$$\gamma_4 = -\frac{D_0\phi_0}{h^4};$$

$$\gamma_5 = \frac{D_0\phi_0}{6h^4};$$

$$\delta_1 = \frac{1}{h}\Lambda_0 - \frac{C_0\phi_0}{h^3} - \frac{A_0\Lambda_0}{2h^5} + \frac{2C_1}{h^5} - C_0 D_0 \frac{\phi_0}{h^5} - \frac{1}{4h^5} C_0\phi_1;$$

$$\delta_2 = -\frac{\Lambda_0}{2h} - \frac{B_0\phi_0}{4h^3} + \frac{B_1}{2h^5} - \frac{B_0}{6h^5}(3D_0\phi_0 + \phi_1) + \frac{1}{2h^5} A_0\Lambda_0;$$

$$\delta_3 = \frac{\Lambda_0}{6h^3};$$

$$\delta_4 = \frac{\Lambda_1}{12h^3} + \frac{2B_1}{3h^5};$$

$$\delta_5 = -\frac{\Lambda_0}{480h^5}.$$

(3) *Stress Intensity Factors*

Expressions for stress intensity factors are of great importance for workers in fracture mechanics. They are given by the relations :

$$\rho = \lim_{x \to 1+} (x-1)^{1/2} [\sigma_y(x, 0)]_{x>1} \qquad \ldots(6.7)$$

$$\tau = \lim_{x \to 1+} (x-1)^{1/2} [\tau_{xy}(x, 0)]_{x>1}. \qquad \ldots(6.8)$$

Putting the values of $\sigma_y(x, 0)$ and $\tau_{xy}(x, 0)$ from (6.5) and (6.6), we get

$$\rho = P_0 [\gamma_1 + 3\gamma_2 + 2\gamma_5] \qquad \ldots(6.9)$$

$$\tau = \tfrac{1}{2} P_0 [\delta_1 + \delta_3]. \qquad \ldots(6.10)$$

7. Numerical Solution of the Integral Equations

The formulae derived in the last sections are of value if the thickness is large as compared to the crack. When its value is only slightly greater than unity, the integral eqns. (4.9) and (4.10) have to be solved numerically. At first sight the solutions would appear to depend only on 't' in the range $0 < t < 1$, but it may be recalled that the kernels $L_1(t, u)$, $L_2(t, u)$, $L_3(t, u)$ and $L_4(t, u)$ defined in Section 4 depend on values of 'h'. The numerical computation of the kernels functions was carried out for the values of $h = 1.05, 1.30, 1.50$. The integral equations were solved by the method of Fox and Goodwin (1953)

The results of calculations of $m(t)$ and $n(t)$ are shown graphically in Figs. 2 and 3. It will be observed that the graphs of $n(t)$ differ very slightly from a straight line. The curves for $m(t)$ are approximately similar to parabolas.

Using these values of $m(t)$ and $n(t)$, normal stress, normal displacement and stress intensity factors were calculated. The results of these calculations are shown in Figs. 4-7.

8. Summary of Results

Expressions for the stresses and displacements in plane of the crack in an elastic incompressible strip have been derived with the aid of solutions of a system of integral equations. Two of these express the condition that the stresses vanish interior to the crack while other two state that in the region exterior to the crack, the displacements vanish.

Although only expressions for one have been derived, the other two stresses are easily found. It is to be noted that as $h \to \infty$, $u(x, 0)$ and $v(x, 0)$ tend to zero.

Again incompressibility assumptions were used primarily because of the resultant simplicity in the elastic field equations. If this is removed, the four integral equations found will be of similar form to eqns. (3.9) to (3.12) and the solution should carry through in the same manner.

Also, we note that if the crack were located symmetrically with respect to the edges of the strip and if the strip were loaded symmetrically, then by using the procedure of Lowengrub on this problem, the Noble's technique reduces to that of Lebdev and Uflyand.

Finally, if the bond is imperfect to the extent that two small cracks exist between strip and foundation is even more interesting. The results are being compiled and will be declared separately.

ACKNOWLEDGEMENT

The authors are greatly indebted to Dr. K. N. Srivastava for many useful discussions.

REFERENCES

Fox, F. G., and Goodwin, E. T. (1953). The numerical solution of non-singular linear integral equations. *Phil. Trans. R. Soc.*, A **245**, 501.

Lebdev, N. N., and Uflyand, Ya. S. (1958). Axisymmetric contact problem for an elastic layer. *J. appl. Math. Mech.*, **22**, 422 (English translation of *Prik Mat. Mech.*, **22**, 320).

Lowengrub, M. (1966). A two dimensional crack problem. *Int. J. Engng. Soc.*, **4**, 289.

Noble, B. (1963). The solution of Bessel function dual integral equations by a multiplying-factor method. *Proc. Camb. phil. Soc.*, **59**, 351.

Sneddon, I. N. (1946). The distribution of stress in the neighbourhood of a crack in an elastic solid. *Proc. R. Soc.*, A **187**, 229.

Tricomi, F. G. (1957). Integral Equations. Interscience Pub., New York.