NON-NULL EIGENVALUES OF SCALAR FIELDS

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Eigenvalues of the massive, the zero-mass and the Brans-Dicke (BD) scalar fields have been obtained. It has been shown that for the massive and the zero-mass fields three of these eigenvalues are equal and the fourth one bears a specific relation with any one of the three equal eigenvalues. In the case of the BD scalar field, however, the eigenvalues are given by a biquadratic relation and, in general, may all be different from one another.

1. INTRODUCTION

In general theory of relativity one of the problems of current interest is the study of 'singularities'. The concept is so obscure that until today no undisputed unanimous view regarding its definition has been accepted. Various authors have proposed different definitions [completeness of geodesics by Geroch (1968), regularity defined through g_{ij} and its derivatives by Bonnor (1954), etc.]. A common view, however, held by a majority of the workers is to study the behaviour of singularities through the invariants of the field. It is, therefore, felt by us that the eigenvalues being such invariants may play a role which might be indicative of the intrinsic feature of singularities.

It may be interesting to note that non-null eigenvalues may exhibit certain characteristic features which might not have been known otherwise. For instance, while studying the nature of singularities through the eigenvalues of the cylindrically symmetric BD Maxwell fields (Rao *et al.* 1978) it has been found that the singularities are exhibited in certain preferred directions only. Such a specific feature is not indicated by any other invariant (e.g., Kretchmann curvature invariant, Ricci scalar etc.) of the field. The occurrence of singularities in certain preferred directions only may appropriately be attributed to and connected with the physical field involved under discussion.

The present work, in view of the above, is an attempt to find the eigenvalues of the scalar fields (the zero-mass field, the massive scalar field and the BD scalar tensor field). It may be remarked here that the relations between the eigenvalues of the physical fields oftentimes help in obtaining the exact solutions under symmetry restrictions prescribed by the line element (Tiwari 1971). Indeed, it is half-way to geometrization of the whole field.

In the case of electromagnetic fields described by the Einstein-Maxwell tensor

$$T_{ij} = \frac{1}{4\pi}\left(-F_{js}F^{s}_{i} + \tfrac{1}{4}\,g_{ij}F_{sp}F^{sp}\right)$$

eigenvalues have been obtained by Lichnerowicz (1955). He has shown that the electromagnetic field possesses eigenvalues which are two to two equal and opposite. Similarly in the case of perfect fluid three of the eigenvalues are equal whereas the fourth one is completely different.

Somewhat similar result has been obtained here in the case of the zero-mass and the massive scalar fields also. It has been found that the scalar fields (mentioned above) possess eigenvalue of which three are equal and the fourth one, although different, bears a specific relation with the equal ones. It is here where a scalar field differs physically from that of perfect fluid (since in the case of perfect fluid the fourth eigenvalue is completely independent of the other three). In the BD theory, however, all the four eigenvalues are obtained from a single biquadratic relation and, in general, may all be different from one another.

2. Eigenvalues of the Massive and the Zero-Mass Scalar Fields

The principal vector x_i, corresponding to a tensor T_{ij} and the associated eigenvalues λ_i, satisfy the relations

$$T^{i}_{j}\,x^{j} = \lambda x^{j}\delta^{i}_{j}\,. \qquad \ldots(1)$$

The eigenvalues are determined from the determinantal equation

$$|\,T_{ij} - \lambda g_{ij}\,| = 0. \qquad \ldots(2)$$

In the case of massive scalar field,

$$T_{ij} = \frac{1}{4\pi}\{v_i v_j - \tfrac{1}{2}\,g_{ij}(v_k v^k - M^2 v^2)\} \qquad \ldots(3)$$

where v is massive scalar field, M the mass associated with it and $v_i = \dfrac{\partial v}{\partial x^i}$.

Substituting (3) in (2) and then simplifying, we get

$$(\lambda + \tfrac{1}{2}P - \tfrac{1}{2}M^2v^2)^3\,(\lambda - \tfrac{1}{2}P - \tfrac{1}{2}M^2v^2) = 0 \qquad \ldots(4)$$

where $P = v^i v_i$. Thus the eigenvalues are given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \tfrac{1}{2}\,(M^2v^2 - P) \qquad \ldots(5)$$

and

$$\lambda_4 = \tfrac{1}{2}(M^2 v^2 + P). \qquad \ldots(6)$$

Consequently in the case of massive scalar field three of the eigenvalues are equal and the sum of any one of these three with the fourth one is always equal to $M^2 v^2$.

When $M = 0$, the energy momentum tensor given by (3) reduces to that of the zero-mass field the eigenvalues for which, as is evident from (5) and (6), are obtained as

$$\lambda_1 = \lambda_2 = \lambda_3 = -\tfrac{1}{2}P \qquad \ldots(7)$$

and

$$\lambda_4 = \tfrac{1}{2}P \qquad \ldots(8)$$

The general property (for the massive scalar field) of the three eigenvalues being equal is as usual true in this case also but now the sum of any one of these with the fourth one is always zero, viz.,

$$\lambda_i + \lambda_4 = 0 \qquad (i = 1, 2, 3). \qquad \ldots(9)$$

3. Eigenvalues of the BD Scalar Fields

The BD vacuum field equations are given by

$$G_{ij} = \frac{w}{v^2}(v_i v_j - \tfrac{1}{2} g_{ij} v_k v^k) + \frac{1}{v}(v_{;ij} - g_{ij} v^k_{;k}) \qquad \ldots(10)$$

and

$$(3 + 2w)\, v^k_{;k} = 0 \qquad \ldots(11)$$

where $G_{ij} = R_{ij} - \tfrac{1}{2} R g_{ij}$, w is the coupling constant and semi-colon denotes covariant derivative with respect to g_{ij}. Equation (11) is satisfied identically even if $w \equiv -3/2$. However, on physical grounds (although it still deserves further consideration) the negative value of w has not yet been accepted and hence we consider the case $v^k_{;k} = 0$ only.

Substituting (10) in (2), and using $v^k_{;k} = 0$ wherever necessary, after simplification, we get

$$\lambda^4 + A\lambda^3 + B\lambda^2 + C\lambda + D = 0, \qquad \ldots(12)$$

where A, B, C and D are given in the Appendix. We thus see that the eigenvalues given by (12), in general, may all be different. In particular cases, however, some of these may be equal (Rao *et al.* 1978).

4. Comparison between Perfect Fluid and Scalar Field

It may be verified that in the case of perfect fluid the eigenvalues are given by

$$\lambda_1 = \lambda_2 = \lambda_3 = -\, p_0 \text{ (pressure)} \qquad \ldots(13)$$

and

$$\lambda_4 = \rho_0 \text{ (density)}. \qquad \ldots(14)$$

Thus, like that of scalar fields, perfect fluid also possesses eigenvalues three of which are equal and the fourth one is completely independent of the other three. Since the two fields (the scalar and the perfect fluid), as regards eigenvalues, exhibit different features with respect to the fourth eigenvalue only this may be taken as the indicative of their physical difference. The study of the fourth eigenvalue, therefore, is very much relevant in distinguishing the two fields physically.

There is an overlaping region, however, when the equation of state for the perfect fluid becomes $\rho_0 = p_0$. Indeed the perfect fluid in this case behaves like a zero-mass field and vice-versa. It may be mentioned here that the same identification has been arrived at by Tabensky and Taub (1973) by a different procedure.

5. Conclusion

As mentioned in the introduction the eigenvalues being one of the many known invariants may be quite useful to study the singular behaviour of physical fields. Interestingly in one of our earlier studies (Rao *et al.* 1978) we have noted that the 'singularities' in the eigenvalues of the fields are exhibited in certain preferred directions only. This suggests that 'the eigenvalue invariants' may explain the physical facts (in certain sense) better as compared to the other known invariants. One therefore needs to know the eigenvalues of various physical fields. Our attempt to find the eigenvalues of the earlier mentioned scalar fields is motivated in view of the above objective as we are mostly engaged in studying the physics of scalar fields.

Eigenvalues are also useful in other respects, particularly in obtaining the exact solutions (Tiwari 1971). At times it also helps in characterizing certain physical situations in a very simple manner. For example, in the case of cylindrically symmetric metric of Marder (1958), the line element is given by

$$ds^2 = e^{2x-2y}(dt^2 - d^2) - \rho^2 e^{-2y} d\Phi^2 - e^{2y+2u} dz^2 \qquad \ldots(15)$$

where x, y, u are functions of ρ and t only. From the determinantal eqn. (2), where T_{ij} corresponds to the zero-mass field, one can see that for the metric (15) two of the eigenvalues are identically equal and the remaining two are given by

$$\lambda^2 - \lambda(T^1_1 + T^4_4) + T^1_1\, T^4_4 - T^1_4\, T^4_1 = 0. \qquad \ldots(16)$$

The condition (9) then gives

$$T_1^1 + T_4^4 = 0 \qquad \text{...(17)}$$

which is equivalent to

$$u_{11} - u_{44} + \frac{2u_1}{\rho} + u_1^2 - u_4^2 = 0. \qquad \text{...(18)}$$

It has already been shown by Marder (1958) that if we take the solution of (18) in the form

$$e^u = f(\rho + t) + g(\rho - t) \qquad \text{...(19)}$$

where f and g are arbitrary functions of their arguments, the third parameter u in the metric (15) can be transformed away and the line element will reduce to that of Einstein-Rosen metric. Hence, in the particular case given above, the general cylindrically symmetric solution of zero-mass field is described by Einstein-Rosen metric.

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APPENDIX

In the following δ^{ijkl}_{mnpq} is the generalized permutation symbol, the latin indices run from 1 to 4 and $P = v^i v_i$.

$$A = -\frac{WP}{v^2}, \; B = -\frac{1}{2v^2} v^i_{;j} v^j_{;i} - \frac{W}{v^3} v^i v_j v^j_{;i}$$

$$C = -\frac{W^3P^3}{4v^6} - \frac{W^2P}{v^5} v^i v_j v^j_{;i} - \frac{WP}{2v^4} v^i_{;j} v^j_{;i}$$

$$-\frac{1}{4}\frac{W^2}{v^4} \delta^{ijkl}_{imnp} v^m v_j v^n_{;k} v^p_{;l} - \frac{1}{4v^4} \delta^{ijkl}_{imnp} v^m_{;j} v^n_{;k} v^p_{;l}$$

$$D = -\frac{W^4P^4}{16\nu^8} - \frac{W^2}{\nu^6}\left(\frac{P^2}{8}\nu^i_{;k}\nu^k_{;i} - \frac{P}{8}\delta^{ijkl}_{imnp}\nu^m\nu_j\nu^n_{;k}\nu^p_{;l}\right)$$

$$+\frac{W}{\nu^5}\left(\delta^{ijkl}_{mnpq}\nu^m\nu_i\nu^n_{;j}\nu^p_{;k}\nu^q_{;l} - \tfrac{1}{4}\delta^{ijkl}_{imnp}\nu^m_{;j}\nu^n_{;k}\nu^p_{;l}\right)$$

$$+\frac{1}{\nu^4}\delta^{ijkl}_{mnpq}\nu^m_{;i}\nu^n_{;j}\nu^p_{;k}\nu^q_{;l}.$$