FORCED OSCILLATIONS OF AN ENCLOSED ROTATING FLUID UNDER A UNIFORM MAGNETIC FIELD

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(*Received* 14 *April* 1978)

In this paper, an attempt is made to examine the properties of forced oscillations in a homogeneous inviscid rotating fluid enclosed in a finite circular cylinder subjected to a uniform magnetic field parallel to the axis of the cylinder. The forced oscillations are generated by subjecting the ends of the cylinder which were initially plane, to simultaneous, identical and axisymmetric time-harmonic oscillations along the axial direction. The solution consisting of a 'periodic' motion which oscillates with forcing frequency together with a doubly infinite set of inertial modes and which is valid for every forcing frequency is found. In contrast to the rotating and stratified fluid cases, the inertial modes whose frequencies form a dense set in the range $(-2\Omega, 2\Omega)$ where Ω is the angular velocity of the rigid rotation, decay with time in the exponential order depending exclusively on the magnetic parameter K. The ultimate motion is periodic, the solution consisting of terms which oscillate with the forcing frequency. Also no resonance occurs in the system and the motion continues to be stable irrespective of the frequency of the inertial modes.

1. INTRODUCTION

Baines (1967) discussed the initial value problem related to axisymmetric forced oscillations of a rigidly rotating inviscid fluid enclosed in a finite circular cylinder using linearized equations. The solution consists of a periodic motion which oscillates with a forcing frequency together with a doubly infinite set of inertial modes. The nature of the periodic solution is strongly dependent on the precise value of the forcing frequency. The system resonates if the forcing frequency coincides with any one of the frequencies of the inertial modes. It was also shown by Baines that no internal sets of discontinuities in velocity or velocity gradient are present in the inviscid flow for finite times. The motion generated by forced oscillations in an inviscid rotating stratified fluid enclosed by a finite cylinder was examined by Devanathan *et al.* (1973) under linearized theory, taking density variation on the inertial terms. The solution consists of a periodic motion which oscillates with the forcing frequency together with numerous inertial-internal modes which propagate energy in the fluid. Also resonance occurs in the system when the forcing frequency is equal to one of the frequencies of internal modes. It was found that there exist some internal modes which grow with time rendering the motion unstable. They concluded that but for the effect of variation of density on the inertial terms the problem of the forced oscillations in stratified and rotating fluids is completely analogous under Boussinesq approximation.

In this paper, an attempt is made to examine the corresponding properties of forced oscillations in a homogeneous inviscid rotating fluid enclosed in a finite circular cylinder subjected to a uniform magnetic field parallel to the axis of the cylinder. The forced oscillations are generated by subjecting the ends of the cylinder which were initially plane, to simultaneous, identical and axisymmetric time-harmonic oscillations along the axial direction. The solution consisting of a 'periodic' motion which oscillates with forcing frequency together with a doubly infinite set of inertial modes and which is valid for every forcing frequency is found. In contrast to the rotating and stratified fluid cases, the inertial modes whose frequencies form a dense set in the range $(-2\Omega, 2\Omega)$ where Ω is the angular velocity of the rigid rotation, decay with time in the exponential order depending exclusively on the magnetic parameter K. The ultimate motion is periodic, the solution consisting of terms which oscillate with the forcing frequency. Also no resonance occurs in the system and the motion continues to be stable irrespective of the frequency of the inertial modes.

2. Governing Equations and Solution

We consider the motion of an inviscid, homogeneous fluid enclosed in a finite circular cylinder of height 2 l and radius a, which is rotating with constant angular velocity Ω about its axis of symmetry. The ends are initially plane and perpendicular to the axis. The fluid which is initially in a state of rigid rotation is under the influence of a uniform magnetic field parallel to the axis of the cylinder. At time $t = 0$ the ends are set impulsively into small time-harmonic oscillations in the axial direction. The deformations are axisymmetric and in phase.

The cylindrical polar coordinates (r, θ, z) are chosen with z-axis along the axis of the cylinder. The ends are initially on the planes $z = \pm 1$. The ends have the motion

$$\left.\begin{aligned} z &= \pm 1 \qquad (t < 0) \\ &= \pm 1 + \epsilon f(r) \sin \alpha t (t \geqslant 0) \end{aligned}\right\} \qquad \ldots(2.1)$$

where $\epsilon \ll 1$, is the frequency of the forced oscillations and $f(r)$ is a regular function of r. Let u, v, w be the velocity components in (r, θ, z) directions respectively. The magnetic Reynolds number is assumed to be much less than unity so as to neglect the induced magnetic field in comparison with the applied magnetic field [Sparrow *et al.* 1962, Verma *et al.* 1969, Bathaiah *et al.* 1975]. Taking the perturbations to be small, the linearized equations of the axisymmetric motion relative to the rotating axis in the absence of any input electric field are

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{\partial p}{\partial r} - Ku \qquad ...(2.2)$$

$$\frac{\partial v}{\partial t} + 2\Omega u = -Kv \qquad ...(2.3)$$

$$\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} \qquad ...(2.4)$$

where $p = \frac{P}{\rho} - \Omega^2 r$ and $K = \frac{\sigma \mu_e^2 H_0^2}{\rho}$, P being the disturbed pressure, ρ the density of the fluid, σ the electrical conductivity, μ_e the magnetic permeability and H_0 the uniform applied magnetic field.

The equation of continuity is

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0. \qquad ...(2.5)$$

Eliminating u, v, w from eqns. (2.2) to (2.4) and using the equation of continuity (2.5), the governing equation for p is

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t} + K\right)\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r}\right) + \left(\frac{\partial^2}{\partial t^2} + 2K\frac{\partial}{\partial t} + K^2 + 4\Omega^2\right)\frac{\partial^2 p}{\partial z^2} = 0. \qquad ...(2.6)$$

The linearized boundary conditions on the fluid at both $z = \pm 1$ are

for $t \geqslant 0$ $\quad w = \epsilon\alpha f(r) \cos \alpha t$ ...(2.7)

and on the cylinder $r = a$

$$u = 0. \qquad ...(2.8)$$

The motion being generated by impulses, initially i.e., when $t = 0$, the velocity vector

$$\bar{q} = -\nabla \pi(r, z)$$

where $\pi(r, z)$ the impulsive pressure function satisfies the Laplace equation

$$\nabla^2 \pi = 0. \qquad ...(2.9)$$

The boundary conditions at $t = 0$ in terms of π are given by

$$\frac{\partial \pi}{\partial r} = 0 \text{ on } r = a \qquad ...(2.10)$$

$$\frac{\partial \pi}{\partial z} = -\epsilon\alpha f(r) \text{ on } z = \pm 1.$$

The solution $\pi(r, z)$ satisfying the Laplace equation (2.9) and conditions (2.10) is

$$\pi(r, z) = -\frac{2\epsilon\alpha}{a^2}\sum_{m=1}^{\infty}\left[\frac{\int_0^a rf(r)\,J_0(k_m r)\,dr}{k_m J_0^2(k_m a)} J_0(k_m r)\frac{\sinh k_m z}{\cosh k_m l}\right]$$

where k_m's are the solutions of $J_1(k_m a) = 0$

By the use of the relations

$$\nabla^2 p - K\frac{\partial w}{\partial z} = 2\Omega\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right)$$

and

$$\frac{\partial}{\partial t}\left(\Delta^2 p - K\frac{\partial w}{\partial z}\right) = 4\Omega^2\frac{\partial w}{\partial z} - 2\Omega K\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right)$$

we have the initial conditions at $t = 0+$

$$\nabla^2 p = -K\frac{\partial^2\pi}{\partial z^2}$$

and

$$\frac{\partial}{\partial t}\left(\nabla^2 p + K\frac{\partial^2\pi}{\partial z^2}\right) = -4\Omega^2\frac{\partial^2\pi}{\partial z^2}$$

which are sufficient to determine p. Also from the above relations we may take

$$p = 0 \text{ at } t = 0.$$

We define the Laplace transform of $p(r, z, t)$ as

$$\bar{p}(r, z, s) = \int_0^{\infty} e^{-st} p(r, z, t)\,dt$$

where $\quad \text{Re}(s) > 0.$

Equation (2.6) in terms of $\bar{p}$ is

$$\frac{\partial^2\bar{p}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{p}}{\partial r} + \lambda_1^2\frac{\partial^2\bar{p}}{\partial z^2} = -\left(\frac{4\Omega^2 + K^2 + Ks}{K^2 + sK}\right)\frac{\partial^2\pi}{\partial z^2} \qquad ...(2.11)$$

where

$$\lambda_1^2 = \left(1 + \frac{4\Omega^2 + K^2 + Ks}{s^2 + Ks}\right).$$

The transformed boundary conditions are

$$\frac{\partial\bar{p}}{\partial r} = 0 \text{ on } r = a \qquad ...(2.12)$$

$$\frac{\partial \bar{p}}{\partial z} = \frac{\epsilon \alpha^3}{\alpha^2 + s^2} f(r) \text{ on } z = \pm 1. \qquad ...(2.13)$$

The general solution of eqn. (2.11) satisfying the boundary conditions (2.12) and (2.13) is

$$\bar{p}(r, z, s) = \frac{2\,\epsilon\alpha}{a^2} \sum_{m=1}^{\infty} \left[\frac{\int_0^a rf(r)\, J_0(k_m r)\, dr}{k_m J_0^2 (k_m a)} J_0(k_m r) \times \left\{ \frac{\sinh k_m z}{\cosh k_m l} - \lambda_1 \left(1 - \frac{\alpha^2}{\alpha^2 + s^2}\right) \frac{\sinh k_m z/\lambda_1}{\cosh k_m l/\lambda_1} \right\} \right]. \qquad ...(2.14)$$

The solution for $p(r, z, t)$ is given by the inverse transform

$$p(r, z, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{p}(r, z, s)\, e^{st}\, ds \qquad (c > 0)$$

$$= \sum_{m=1}^{\infty} C_m J_0(k_m r) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[\frac{\sinh k_m z}{\cosh k_m l} - \lambda_1 \left(1 - \frac{\alpha^2}{\alpha^2 + s^2}\right) \frac{\sinh k_m z/\lambda_1}{\cosh k_m l/\lambda_1} \right] e^{st}\, ds \qquad ...(2.15)$$

where

$$C_m = \frac{2\,\epsilon\alpha}{a^2} \frac{\int_0^a rf(r)\, J_0(k_m r)\, dr}{k_m J_0^2 (k_m a)}.$$

The above integrand has simple poles at

$$s = \pm\, i\alpha \text{ and } s = -\, K\delta_{mn} \pm \beta_{mn} \quad (\text{for } n \leqslant N)$$

and

$$s = -\, K\delta_{mn} \pm i\beta^*_{mn} \qquad (\text{for } n > N)$$

where m is a positive integer, n an integer,

$$\delta_{mn} = \frac{k_m^2\, l^2 + 2(2n+1)^2\, \pi^2}{k_m^2\, l^2 + (2n+1)^2\, \pi^2}$$

$$\beta_{mn} = \frac{2(2n+1)\,\pi\, [k_m^2\, l^2 k^2 - 4\Omega^2 \{k_m^2\, l^2 + (2n+1)^2\, \pi^2\}]^{1/2}}{k_m^2\, l^2 + (2n+1)^2\, \pi^2}$$

$$\beta^*_{mn} = \frac{2(2n+1)\pi\,[4\Omega^2\{k_m^2 l^2 + (2n+1)^2\pi^2\} - k_m^2 l^2 k^2]^{1/2}}{k_m^2 l^2 + (2n+1)^2\pi^2}$$

N is such that $1 + \dfrac{(2n+1)^2\pi^2}{k_m^2 l^2} < \dfrac{k^2}{4\Omega^2}$

for $n \leqslant N$, for a fixed k_m.

and

$$1 + \frac{(2n+1)^2\pi^2}{k_m^2 l^2} > \frac{k^2}{4\Omega^2}$$

for $n > N$, for a fixed k_m.

Clearly $|-K\delta_{mn} \pm \beta_{mn}|$, $|-K\delta_{mn} \pm i\beta^*_{mn}|$

are less than $(4\Omega^2 + K^2)^{1/2}$ for all m, n and the set of poles $(-K\delta_{mn} \pm i\beta^*_{mn})^{1/2}$ for each m has limit points at $-K \pm 2i\Omega$. The function has an essential singularity at each of these two points. The inversion integral in (2.15) is evaluated in the usual method of residues and thus $p(r, z, t)$ is found to be

$$p(r, z, t) = \sum_{m=1}^{\infty} C_m J_0(k_m r)\Big[\tfrac{1}{2}\alpha^{1/2}\{(\mu P_m - \lambda Q_m)\cos\alpha t$$

$$+ (\lambda P_m + \mu Q_m)\sin\alpha t\} + \sum_{n=0}^{N}\Big\{X_{mn}\frac{\sinh A_{mn}z}{\sinh A_{mn}l}\exp\frac{t}{2}$$

$$(-\beta_{mn} - K\delta_{mn}) + \bar{X}_{mn}\frac{\sinh \bar{A}_{mn}z}{\sinh \bar{A}_{mn}l}\exp\frac{t}{2}(\beta_{mn} - K\delta_{mn})\Big\}$$

$$+ \sum_{n=N+1}^{\infty} \mathrm{Re}\Big\{X^*_{mn}\frac{\sinh \overset{*}{A}_{mn}z}{\sinh A^*_{mn}l}\exp\frac{t}{2}(-i\beta^*_{mn} - K\delta_{mn})$$

$$+ \bar{X}^*_{mn}\bar{X}_{mn}\frac{\sinh \bar{A}^*_{mn}z}{\sinh \bar{A}^*_{mn}l}\exp\frac{t}{2}(i\beta^*_{mn} - K\delta_{mn})\Big\}\Big] \quad \text{...(2.16)}$$

where

$$\lambda = \left[\frac{\alpha^2(4\Omega^2 - K^2 - \alpha^2)^2 + K^2(4\Omega^2 + K^2 + \alpha^2)^2}{(\alpha^2 + K^2)^2}\right]^{1/4}$$

$$\times \cos\left\{\tfrac{1}{2}\tan^{-1}\frac{K(4\Omega^2 + K^2 + \alpha^2)}{\alpha(4\Omega^2 - K^2 - \alpha^2)}\right\}.$$

$$\mu = \left[\frac{\alpha^2(4\Omega^2 - K^2 - \alpha^2)^2 + K^2(4\Omega^2 + K^2 + \alpha^2)^2}{(\alpha^2 + K^2)^2}\right]^{1/4}$$
$$\times \sin\left\{\tfrac{1}{2}\tan^{-1}\frac{K(4\Omega^2 + K^2 + \alpha^2)}{\alpha(4\Omega^2 - K^2 - \alpha^2)}\right\}.$$

$$R_m P_m = \sin a_{m\lambda} z \,.\, \cosh a_{m\mu} l \,.\, \cos a_{m\lambda} l \,.\, \cosh a_{m\mu} z$$
$$- \sinh a_{m\mu} z \,.\, \sinh a_{m\mu} l \,.\, \sin a_{m\lambda} l \,.\, \cos a_{m\lambda} z.$$

$$R_m Q_m = \sin a_{m\lambda} z \,.\, \cosh a_{m\mu} z \,.\, \sin a_{m\lambda} l \,.\, \sinh a_{m\mu} l$$
$$+ \sinh a_{m\mu} z \,.\, \cos a_{m\lambda} z \,.\, \cos a_{m\lambda} l \,.\, \cosh a_{m\mu} l.$$

$$R_m = \cosh^2 a_{m\mu} l \,.\, \cos^2 a_{m\lambda} l + \sinh^2 a_{m\mu} l \sin^2 a_{m\lambda} l.$$

$$a_{m\lambda} = \frac{\lambda k_m \alpha^{1/2}}{\lambda^2 + \mu^2}, \quad a_{m\mu} = \frac{\mu k_m \alpha^{1/2}}{\lambda^2 + \mu^2}$$

$$X_{mn} = [(\beta_{mn} + K\delta_{mn})^2 \{16\Omega^2 + (2K - \beta_{mn} - K\delta_{mn})^2\}]$$
$$\div [2k_m l \{4\alpha^2 + (\beta_{mn} + K\delta_{mn})^2\}$$
$$\times \{16\Omega^2(K - \beta_{mn} - K\delta_{mn}) + K(2K - \beta_{mn} - K\delta_{mn})^2\}].$$

$$A_{mn} = k_m [(\beta_{mn} + k\delta_{mn})(\beta_{mn} + K\delta_{mn} - 2K)$$
$$\div \{16\Omega^2 + (2K - \beta_{mn} - K\delta_{mn})^2\}]^{1/2}.$$

$\bar{X}_{mn}$ and $\bar{A}_{mn}$ are obtained from X_{mn} and A_{mn} respectively by changing the sign of β_{mn}; X^*_{mn} and A^*_{mn} are obtained from X_{mn} and A_{mn} respectively by replacing β_{mn} by $i\beta^*_{mn}$; and $\bar{X}^*_{mn}$ and $\bar{A}^*_{mn}$ are obtained from X^*_{mn} and A^*_{mn} respectively by changing the sign of β^*_{mn}.

The above solution for p is true for all α, the frequency of the forced oscillations. Using the eqns. (2.2) to (2.4), the velocity components u, v, w in terms of p are given by the expressions

$$u = \frac{1}{4\Omega^2(4\Omega^2 + K^2) r} \int^r r\Big[(K\nabla^2 p_{tt} - (4\Omega^2 - K^2)\nabla^2 p_t$$
$$- 4\Omega^2 K \nabla^2 p + K^2 \frac{\partial^2 p_t}{\partial z^2} + K^3 \frac{\partial^2 p}{\partial z^2}\Big] dr.$$

$$v = -\frac{1}{2\Omega(4\Omega^2 + K^2) r} \int^r r\left[K\nabla^2 p_t - 4\Omega^2 \nabla^2 p + K^2 \frac{\partial^2 p}{\partial z^2}\right] dr. \qquad \ldots(2.17)$$

$$w = \frac{1}{4\Omega^2 + K^2} \int^z \left(\nabla^2 p_t + K\nabla^2 p + K\frac{\partial^2 p}{\partial z^2}\right) dz$$

where $p_{tt} = \dfrac{\partial^2 p}{\partial t^2}$, $p_t = \dfrac{\partial p}{\partial t}$. It is evident that the velocities are linear functions of the derivatives of the disturbed pressure p. We now consider the consequences of the solution for p. It has the form

$$p(r, z, t) = \sum_{m=1}^{\infty} C_m J_0(k_m r) \tfrac{1}{2} \alpha^{1/2} \{(\mu p_m - \lambda Q_m) \cos \alpha t + (\lambda P_m + \mu Q_m) \sin \alpha t\} + \text{inertial modes.} \quad \ldots(2.18)$$

3. Conclusions

The expressions $\mu P_m - \lambda Q_m$, $\lambda P_m + \mu Q_m$ in (2.18) contain both the terms of the type $\sin a_{m\lambda} z$, $\cos a_{m\lambda} z$, $\cosh a_{m\mu} z$, $\sinh a_{m\mu} z$, which indicate that the motion prevails everywhere in the domain, unlike the case of rotating fluids wherein the motion is near the ends of the cylinder with exponential decay towards the centre. The inertial modes consist of free wave modes with frequencies β^*_{mn} which form a dense set in the range $(-2\Omega, 2\Omega)$. In contrast to the rotating and stratified cases these inertial modes are no longer persistant and they decay with time in the order of $\exp\left(-\dfrac{t}{2} K\delta_{mn}\right)$. Also the system never resonates for any frequency of the forced oscillations and the inertial modes decay exponentially with time rendering the motion always stable once again in contrast to either the rotating or the stratified fluid motion wherein resonance occurs when the forcing frequency coincides with one of the frequencies of the inertial modes and wherein there exist some modes which grow with time making the motion unstable. The singularity $\alpha = 2\Omega$ obtained by Baines (1967) in his paper was eliminated by the magnetic field. It is well known that the magnetic lines of force act as taut lines preventing occurrence of singularities in the field.

Acknowledgement

The author is thankful to Professor R. Seetharamaswamy for his valuable guidance during the preparation of this paper.

References

Baines, P. G. (1967). Forced oscillations of an enclosed rotating fluid. *J. Fluid Mech.*, **30** (3), 533.

Bathaiah, D., Krishna, D. V., and Seetharamaswamy, R. (1975). Stratified rotating viscous flow between two disks under transverse magnetic field. *Proc. Indian Acad. Sci.*, **82A** (1), 17.

Devanathan, R., and Rao, A. R. (1973). Oscillations of an enclosed rotating stratified fluid. *ZAMM*, **53**, 617.

Sparrow, E. M., and Cess, R. D. (1962). Magnetohydrodynamic flow and heat transfer about a rotating disk. *Trans. ASME, J. appl. Mech.*, **29** (1), 181.

Verma, P. D., and Mathur, A. K. (1969). Magnetohydrodynamic flow between two parallel plates, one in uniform motion and the other at rest with uniform suction at stationary plate. *Proc. natn. Inst. Sci. India*, **35**, 507.