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A NOTE ON AN APPLICATION OF A FIXED POINT THEOREM IN APPROXIMATION THEORY

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We extend a theorem of Singh [*J. approx. Theory* 25 (1979) 89] by relaxing further the condition of Star-shapedness on the set of best C approximants to a given point.

Singh[3] proved a theorem on existence of T-invariant point on a normed linear space X. That gave an extension of a result of Brosowski[1] given for a contractive linear operator T. We extend the result of Singh[3] further in this note. First we state the result of Singh[3].

Theorem 1—Let T be a contractive operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If the set of best C-approximants to x is nonempty, compact and starshaped, then it contains a T-invariant point. We need the following preliminaries.

Definition 1—A mapping $T : X \to X$ is said to be nonexpansive if $\| Tx - Ty \| \leqslant \| x - y \|$, for all $x, y \in X$.

Definition 2—A family of maps $\{f_\alpha\}_{\alpha \in X}$ is said to be an (S)-convex structure on X if it satisfies the following conditions :

(i) $f_\alpha : [0, 1] \to X$, i.e., f_α is a map from $[0, 1]$ into X for each $\alpha \in X$.

(ii) $f_\alpha (1) = \alpha$ for each $\alpha \in X$.

(iii) $f_\alpha (t)$ is jointly continuous in (α, t) i.e. $f_\alpha (t) \to f_{\alpha_0} (t_0)$ for $\alpha \to \alpha_0$ in X and $t \to t_0$ in $[0, 1]$.

(iv) If T is a map from X into itself, then for any $x \in X, f_{Tx} (t) \subset Tx$, for all $t \in [0, 1]$.

(v) $\| f_\alpha (t) - f_\beta (t) \| \leqslant \phi (t) \| \alpha - \beta \|$ where ϕ is a function from $[0, 1)$ into itself. Our extension is the following :

Theorem 2—Let T be a nonexpansive mapping on normed linear space X. Let there exists an (S)-convex structure on X. Let C be a T-invariant subset of X and x a T-invariant point. If the set of best C-approximants to x is nonempty and compact, then it contains a T-invariant point.

PROOF : Let D be the set of best C-approximants to x. Then $T : D \to D$ (since, if $y \in D$, then $\| Ty - x \| = \| Ty - Tx \| \leqslant \| y - x \|$, then $Ty \in D$). Let k_n, $0 \leqslant k_n < 1$, be a sequence of real numbers such that $k_n \to 1$ as $n \to \infty$. Then define T_n as $T_n(x) = f_{Tx}(k_n)$ for $x \in D$. It is easy to see because of (iv) in Definition (2) that T_n also maps D into D for each n. Also we have, because of (v) in Definition 2,

$$\| T_n x - T_n y \| = \| f_{Tx}(k_n) - f_{Ty}(k_n) \|$$

$$\leqslant \phi(k_n) \|Tx - Ty\|$$

$$\leqslant \phi(k_n) \| x - y \|.$$

From Dotson[2] we see that each T_n is a contraction and hence has a unique fixed point, say x_n for each n. Thus $T_n x_n = x_n$ for each n. Since D is compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_i}\}$ converging to $\bar{x}$ say. We claim that $T\bar{x} = \bar{x}$. Now, as $n_i \to \infty$, $x_{n_i} = T_{n_i} x_{n_i} = f_{Tx_{n_i}}(k_{n_i}) \to f_{T\bar{x}}(1) = T\bar{x}$, since f_α is jointly continuous and T is nonexpansive (and therefore continuous and $Tx_{n_i} \to T\bar{x}$ as $n_i \to \infty$). From which we get $\bar{x} = T\bar{x}$. Thus $\bar{x}$ is a T-invariant.

REFERENCES

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3. S. P. Singh, *J. Approx. Theory* **25** (1979) 89–90.