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# SOME PROPERTIES OF FRACTIONAL INTEGRALS AND DERIVATIVES OF UNIVALENT FUNCTIONS

G. LAKSHMA REDDY AND K. S. PADMANABHAN

*Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Madras* 600005

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The fractional derivative of order $\alpha$ of an analytic function $f$ is defined by

$$D_z^{\alpha} f(z) = \frac{1}{(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt, \; 0 < \alpha < 1$$

and the fractional integral of order $\alpha$ is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt, \; \alpha > 0.$$

In this paper some properties of fractional integrals and derivatives of $f$ are discussed when $f$ lies in different sub-classes of starlike and convex functions in the unit disk.

## 1. INTRODUCTION

Let $P(A, B)$ denote the class of analytic functions $p$ defined in the unit disc $E = \{z : |z| < 1\}$ having the representation

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \; -1 \leqslant B < A \leqslant 1, \; z \in E$$

where $w$ is analytic in $E$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$. We define the classes $S(A, B)$ and $K(A, B)$ as follows :

$$S(A, B) = \left\{ f : \frac{zf'(z)}{f(z)} \in P(A, B) \right\}$$

and

$$K(A, B) = \{f : zf'(z) \in S(A, B)\}.$$

It immediately follows that $p(z) \prec \frac{1+Az}{1+Bz}$, $z \in E$, where $\prec$ denotes subordination.

We need the following definitions given by Owa[2].

*Definition* 1—Fractional derivative of order $\alpha$ of an analytic function $f$ is defined by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} dt$$

where $0 < \alpha < 1$, $f$ is an analytic function in a simply connected region of the $z$-plane containing the origin, and the multiplicity of $(z-t)^{-\alpha}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)$ is greater than zero. Clearly

$$f(z) = \lim_{\alpha \to 0} D_z^{\alpha} f(z)$$

and

$$f'(z) = \lim_{\alpha \to 1} D_z^{\alpha} f(z).$$

*Definition* 2—The fractional integral of order $\alpha$ is defined by

$$D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(t)}{(z-t)^{1-\alpha}} dt$$

where $\alpha > 0$, $f$ is an analytic function in a simply connected region of the $z$-plane containing the origin, and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)$ is greater than zero.

Clearly $f(z) = \lim_{\alpha \to 0} D_z^{-\alpha} f(z)$.

The fractional derivative of order $(n+\alpha)$ is defined by

$$D_z^{n+\alpha} f(z) = \frac{d^n}{dz^n} D_z^{\alpha} f(z)$$

where $0 < \alpha < 1$ and $n = 0, 1, 2, \ldots$.

For the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $E$ we put

$$F(z) = \Gamma(2+\alpha)\, z^{\alpha} D_z^{-\alpha} f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{n!\,\Gamma(2+\alpha)}{\Gamma(n+1+\alpha)} a_n z^n, \alpha > 0 \qquad \ldots(1)$$

and

$$G(z) = \Gamma(2-\alpha)\, z^{\alpha}\, D_z^{\alpha} f(z)$$

$$= z + \sum_{n=2}^{\infty} \frac{n!\,\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\, a_n z^n,\ 0 < \alpha < 1. \qquad \ldots(2)$$

## 2. Some Properties of Fractional Integrals and Derivatives

*Theorem* 1—There exist univalent starlike functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ in } S(A, B) \text{ such that } F \in S(A, B).$$

Proof : Let $f(z) = z + a_2 z^2 \in S(A, B)$, $a_2$ positive. Then

$$\frac{zf'(z)}{f(z)} = \frac{1 + 2a_2 z}{1 + a_2 z} \prec \frac{1 + Az}{1 + Bz},\ -1 \leqslant B < A \leqslant 1.$$

This implies that

$$(1 + Ar)/(1 + Br) \geqslant (1 + 2a_2 r)/(1 + a_2 r) \text{ for all } r < 1 \qquad \ldots(3)$$

and

$$(1 - Ar)/(1 - Br) \leqslant (1 - 2a_2 r)/(1 - a_2 r) \text{ for all } r < 1. \qquad \ldots(4)$$

Now (3) implies that $a_2\{1 + r(2B - A)\} \leqslant A - B$ for all $r$. If $2B \geqslant A$, this will be satisfied only if $a_2 \leqslant (A - B)/(2B - A + 1)$. If $2B \leqslant A$, the above inequality will be satisfied only if $a_2 \leqslant A - B$. So we get

$$\begin{cases} \text{(a) } a_2 \leqslant (A - B)/(2B + 1 - A) \text{ if } 2B \geqslant A \\ \text{(b) } a_2 \leqslant A - B \text{ if } A \geqslant 2B. \end{cases} \qquad \ldots(5)$$

Similarly (4) implies that

$$\begin{cases} \text{(a) } a_2 \leqslant (A - B)/(A - 2B + 1) \text{ if } A \geqslant 2B \\ \text{(b) } a_2 \leqslant A - B \text{ if } A \leqslant 2B. \end{cases} \qquad \ldots(6)$$

(5) and (6) should hold simultaneously. We now consider

*Case i*—Let $A \leqslant 2B$ so that $2B - A \geqslant 0$. Then (5a) and (6b) hold. But (5a) $\Rightarrow$ (6b).

*Case ii*—Let $A \geqslant 2B$ so that (5b) and (6a) hold. But (6a) $\Rightarrow$ (5b).

For $f(z) = z + a_2 z^2$, $(a_2 > 0)$ we observe that $F(z) = z + \frac{2}{2+\alpha} a_2 z^2$, $\alpha > 0$.

$zF'(z)/F(z) = \dfrac{1+\frac{4a_2z}{2+\alpha}}{1+\frac{2a_2z}{2+\alpha}}$ maps $|z| \leqslant r$ onto the disc as the segment joining $\left(1-\frac{4a_2}{2+\alpha}r\right)\Big/\left(1-\frac{2a_2r}{2+\alpha}\right)$, $\left(1+\frac{4a_2r}{2+\alpha}\right)\Big/\left(1+\frac{2a_2r}{2+\alpha}\right)$ as diameter. We shall show that $zF'(z)/F(z) \prec (1+Az)/(1+Bz)$. To this end we have only to check the following inequalities.

$$\begin{cases} \text{(a)}\ \dfrac{1-Ar}{1-Br} \leqslant \dfrac{1-A_1r}{1-B_1r} \\ \text{(b)}\ \dfrac{1+A_1r}{1+B_1r} \leqslant \dfrac{1+Ar}{1+Br} \end{cases} \quad \text{for all } r \qquad \ldots(7)$$

where $A_1 = 4a_2/(2+\alpha)$ and $B_1 = 2a_2/(2+\alpha)$.

Now (7a) is true if $A-B \geqslant 2a_2(r(A-2B)+1)/(2+\alpha)$ for all $r$ which is in turn will be satisfied if

$$\begin{cases} A-B \geqslant 2a_2(1+A-2B)/(2+\alpha) \text{ which } A \geqslant 2B \\ A-B \geqslant 2a_2/(2+\alpha) \text{ when } A \leqslant 2B. \end{cases} \qquad \ldots(8)$$

Similarly (7b) will hold provided $A-B \geqslant 2a_2(1+r(2B-A))/(2+\alpha)$ for all $r$, which in turn will be true if

$$\begin{cases} A-B \geqslant 2a_2(1+2B-A)/(2+\alpha) \text{ for } A \leqslant 2B \\ A-B \geqslant 2a_2/(2+\alpha) \text{ for } A \geqslant 2B. \end{cases} \qquad \ldots(9)$$

Comparing (8) and (9) for the cases $A \leqslant 2B$ and $A \geqslant 2B$ shows that (7a) and (7b) will hold simultaneously for all $r < 1$ provided

$$\begin{cases} \text{(a)}\ A-B \geqslant 2a_2(1+2B-A)/(2+\alpha) \text{ for } A \leqslant 2B \\ \text{(b)}\ A-B \geqslant 2a_2(1+A-2B)/(2+\alpha) \text{ for } A \geqslant 2B. \end{cases} \qquad \ldots(10)$$

But these two inequalities follow from (5b) and (6a) respectively.

*Corollary* 1—If $f(z) = z + a_2z^2$, $a_2 > 0$, belongs to $S(A, B)$, then $F(z)$ is an univalent convex function in (i) $|z| < (2+\alpha)(2B-A+1)/8(A-B)$ if $A \leqslant 2B$ (ii) $|z| < (2+\alpha)(A-2B+1)/8(A-B)$ if $2B \leqslant A$, where $\alpha$ is a positive real number.

PROOF : Since $f(z) = z + a_2z^2$, we write $F$ as $F(z) = z + 2a_2z^2/(2+\alpha)$. Then

$$|zF''(z)/F'(z)| = |(4a_2z/(2+\alpha))/(1+4a_2z/(2+\alpha))| < 1$$

if

$$| 4a_2z/(2 + \alpha) | < | 1 + 4a_2z/(2 + \alpha) |$$

or if

$$| 4a_2z/(2 + \alpha) |^2 < 1 + | 4a_2z/(2 + \alpha) |^2 + (8a_2/(2 + \alpha)) \operatorname{Re} z$$

which holds if $\operatorname{Re} z > -(2 + \alpha)/8a_2$.

(i) If $A \leqslant 2B$, since $f \in S(A, B)$, $a_2 \leqslant (A - B)/(2B - A + 1)$. Now $\operatorname{Re} z > -(2 + \alpha)/8a_2$ is true if $\operatorname{Re} z > -(2 + \alpha)(2B - A + 1)/8(A - B)$, which in turn is true if $| z | < (2 + \alpha)(2B - A + 1)/8(A - B)$.

(ii) If $2B \leqslant A$, then $a_2 \leqslant (A - B)/(A - 2B + 1)$. Thus $\operatorname{Re} z > -(2 + \alpha)/8a_2$ is true if, $\operatorname{Re} z > -(2 + \alpha)(A - 2B + 1)/8(A - B)$. Hence if $2B \leqslant A$, $F$ is convex in $| z | < (2 + \alpha)(A - 2B + 1)/8(A - B)$.

*Theorem* 2—There exist functions $f \in K(A, B)$ such that $F \in K(A, B)$.

PROOF: Let $f(z) = z + a_2 z^2 \in K(A, B)$, $a_2$ positive, $-1 \leqslant B < A \leqslant 1$. Then $1 + zf''(z)/f'(z) = (1 + 4a_2z)/(1 + 2a_2z) \prec (1 + Az)/(1 + Bz)$. This implies that

$$(1 + Ar)/(1 + Br) \geqslant (1 + 4a_2r)/(1 + 2a_2r) \qquad \ldots(11)$$

and

$$(1 - Ar)/(1 - Br) \leqslant (1 - 4a_2r)/(1 - 2a_2r). \qquad \ldots(12)$$

Now (11) implies that $2a_2(2B - A + 1) \leqslant A - B$ which is trivial if $2B - A + 1 \leqslant 0$. If $2B - A + 1 > 0$ then (11) implies that

$$a_2 \leqslant (A - B)/2(2B - A + 1). \qquad \ldots(13)$$

Again (12) implies that

$$a_2 \leqslant (A - B)/2(A - 2B + 1). \qquad \ldots(14)$$

To this end it is enough to verify the following inequalities $(1 - Ar)/(1 - Br) \leqslant (1 - A_1r)/(1 - B_1r)$ and $(1 + A_1r)/(1 + B_1r) \leqslant (1 + Ar)/(1 + Br)$ where $A_1 = 8a_2/(2 + \alpha)$ and $B_1 = 4a_2/(2 + \alpha)$.

The remaining part of the proof follows on the same lines as in Theorem 1, and hence is omitted. We just remark that (13) holds for $A \leqslant 2B$ and implies (14). For $2B + 1 \geqslant A \geqslant 2B$, (14) holds and implies (13). For $A \geqslant 2B + 1$, (14) alone holds.

*Theorem* 3—There exist functions $f \in K(A, B)$ such that $G \in S(A, B)$.

PROOF: Let $f(z) = z + a_2 z^2 \in K(A, B)$, $a_2$ positive. Since

$$G(z) = z + 2a_2 z^2/(2 - \alpha), \quad 0 < \alpha < 1.$$

$zG'(z)/G(z) = (1 + 4a_2 z/(2-\alpha))/(1 + 4a_2 z/(2-\alpha))$ maps $|z| \leqslant r$ onto the disc as the segment joining $(1 - 4a_2 r/(2-\alpha))/(1 - 2a_2 r/(2-\alpha))$, $(1 + 4a_2 r/(2-\alpha))/(1 + 2a_2 r/(2-\alpha))$ as diameter. We need only verify that $zG'(z)/G(z) \prec (1 + Az)/(1 + Bz)$. Now we have to prove the following two inequalities.

$$(1 - Ar)/(1 - Br) \leqslant (1 - A_1 r)/(1 - B_1 r)$$

and

$$(1 + A_1 r)/(1 + B_1 r) \leqslant (1 + Ar)/(1 + Br)$$

where $A_1 = 4a_2/(2-\alpha)$ and $B_1 = 2a_2/(2-\alpha)$.

This is done by using (13) and (14) as the case may be, as in the Proof of Theorem 1.

*Corollary* 2—If $f(z) = z + a_2 z^2$ belongs to $K(A, B)$, then $G(z)$ is an univalent convex function in

(i) $|z| < (2-\alpha)(2B - A - 1)/4(A - B)$ for $A \leqslant 2B$.

(ii) $|z| < (2-\alpha)(A - 2B + 1)/4(A - B)$ for $2B \leqslant A$

where $0 < \alpha < 1$.

Proof is similar to that of Corollary 1.

*Corollary* 3—If $f(z) = z + a_2 z^2$ belongs to $S(A, B)$, then $G(z)$ is an univalent convex function in (i) $|z| < (2-\alpha)(2B - A + 1)/8(A - B)$ for $A \leqslant 2B$ and (ii) $|z| < (2-\alpha)(A - 2B + 1)/8(A - B)$ for $2B \leqslant A$, where $0 < \alpha < 1$.

*Corollary* 4—If $f(z) = z + a_2 z^2$ belongs to $S(A, B)$, then $G(z)$ is an univalent starlike function in (i) $|z|(2-\alpha)(2B - A + 1)/4(A - B)$ for $A \leqslant 2B$ and (ii) $|z| < (2-\alpha)(A - 2B + 1)/4(A - B)$ for $2B \leqslant A$, where $0 < \alpha < 1$.

## 3. The Classes $S_F(A, B)$, $K_F(A, B)$ and $S_G(A, B)$

We now define the following classes of functions. Let $S_F(A, B)$ denote the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ for which $F(z) \in S(A, B)$, $K_F(A, B)$ denote the class of functions $f$ for which $F(z) \in K(A, B)$ and let $S_G(A, B)$ denote the class of functions $f$ for which $G(z) \in S(A, B)$. [Here $f$, $F$, $G$ are related as in (1) and (2)]. We establish distortion theorems for the above mentioned classes.

*Theorem* 4—Let $f \in S_F(A, B)$. Then for $|z| < r < 1$

(a) $$\frac{r^{1+\alpha}(1-Br)^{(A-B)/B}}{\Gamma(2+\alpha)} \leqslant |D_z^{-\alpha} f(z)| \leqslant \frac{r^{1+\alpha}(1+Br)^{(A-B)/B}}{\Gamma(2+\alpha)}, \text{ for } B \neq 0$$

(b) $$\frac{r^{1+\alpha} e^{-Ar}}{\Gamma(2+\alpha)} \leqslant |D_z^{-\alpha} f(z)| \leqslant \frac{r^{1+\alpha} e^{Ar}}{\Gamma(2+\alpha)}, \text{ for } B = 0$$

where $\alpha$ is a positive real number and $-1 \leqslant B < A \leqslant 1$.

PROOF : Since $F \in S(A, B)$, then by Theorem 4[1] we have

$$r(1 - Br)^{(A-B)/B} \leqslant |F(z)| \leqslant r(1 + Br)^{(A-B)/B} \text{ for } B \neq 0$$

and

$$re^{-Ar} \leqslant |F(z)| \leqslant r e^{Ar} \text{ for } B = 0.$$

Hence the required results (a) and (b) follow by using the definition of $F(z)$ (1). Similarly we prove the following

*Theorem* 5—Let $f \in S_G(A, B)$. Then for $|z| < r < 1$

$$\frac{r^{1-\alpha}(1-Br)^{(A-B)/B}}{\Gamma(2-\alpha)} \leqslant |D_z^{\alpha} f(z)| \leqslant \frac{r^{1-\alpha}(1+Br)^{(A-B)/B}}{\Gamma(2-\alpha)} \text{ for } B \neq 0$$

and

$$\frac{r^{1-\alpha} e^{-Ar}}{\Gamma(2-\alpha)} \leqslant |D_z^{\alpha} f(z)| \leqslant \frac{r^{1-\alpha} e^{Ar}}{\Gamma(2-\alpha)} \text{ for } B = 0$$

where $0 < \alpha < 1$ and $-1 \leqslant B < A \leqslant 1$.

*Theorem* 6—Let $f \in K_F(A, B)$. Then for $|z| < r < 1$

(a) $$\frac{r^{\alpha}\{1-(1-Br)^{A/B}\}}{A\Gamma(2+\alpha)} \leqslant |D_z^{-\alpha} f(z)| \leqslant \frac{r^{\alpha}\{(1+Br)^{A/B}-1\}}{A\Gamma(2+\alpha)} \text{ for } B \neq 0$$

(b) $$r^{\alpha}(1-e^{-Ar})/A\,\Gamma(2+\alpha) \leqslant |D_z^{-\alpha} f(z)| \leqslant r^{\alpha}(e^{Ar}-1)/A\Gamma(2+\alpha)$$

for $B = 0$ where $\alpha$ is a positive real number.

We require the following

*Lemma*—Let $f \in K(A, B)$. Then for $|z| < r < 1$

$$\frac{1}{A}\{1-(1-Br)^{A/B}\} \leqslant |f(z)| \leqslant \frac{1}{A}\{(1+Br)^{A/B}-1\} \text{ for } B \neq 0$$

and

$$\frac{1}{A}(1-e^{-Ar}) \leqslant |f(z)| \leqslant \frac{1}{A}(e^{Ar}-1).$$

PROOF : Since $f \in K(A, B)$ implies $zf'(z) \in S(A, B)$, by Theorem 4[1]. We have

$$r(1-Br)^{(A-B)/B} \leqslant |zf'(z)| \leqslant r(1+Br)^{(A-B)/B},$$

That is

$$(1-Br)^{(A-B)/B} \leqslant |f'(z)| \leqslant (1+Br)^{(A-B)/B} \text{ for } B \neq 0.$$

and

$$e^{-Ar} \leqslant |f'(z)| \leqslant e^{Ar} \text{ for } B = 0.$$

Now

$$f(z) = \int_0^z f'(u)\,du = \int_0^r e^{i\phi} f'(te^{i\phi})\,dt,\ u = te^{i\phi},\ 0 \leqslant t < 1.$$

$$|f(z)| \leqslant \int_0^r |f'(te^{i\phi})|\,dt \leqslant \int_0^r (1+Bt)^{(A-B)/B}\,dt$$

$$= \frac{1}{A}\{(1+Br)^{A/B} - 1\} \text{ for } B \neq 0.$$

For $B = 0$ we have

$$|f(z)| \leqslant \int_0^r |f'(te^{i\phi})|\,dt \leqslant \int_0^r e^{At}\,dt = \frac{1}{A}(e^{Ar} - 1).$$

Again

$$\text{Re } r\,\frac{\partial}{\partial r}\log f'(z) = \text{Re } zf''(z)/f'(z) \geqslant (B-A)\,r/(1-Br)$$

since $\text{Re}\{1 + zf''(z)/f'(z)\} \geqslant (1-Ar)/(1-Br)$. So

$$\frac{\partial}{\partial r}\log|f'(z)| \geqslant (B-A)/(1-Br).$$

On integration we get

$$|f'(z)| \geqslant (1-Br)^{(A-B)/B}.$$

Let $f(re^{i\theta}) = \text{Re}^{i\phi}$. The straight line segment $L$ from 0 to $\text{Re}^{i\phi}$ lies entirely in the image of $E$ by $f$, since $f \in K(A, B)$. $L$ corresponds to a path $l$ in $E$ which joins 0 to $re^{i\theta}$. So we have

$$R = \int_L |dw| = \int_l \left|\frac{dw}{dz}\right| |dz| \geqslant \int_l (1-Br)^{(A-B)/B}\,dt = \frac{1}{A}\{1-(1-Br)^{A/B}\}$$

for $B \neq 0$. For $B = 0$,

$$R = \int_L |dw| = \int_l \left|\frac{dw}{dz}\right| |dz| \geqslant \int_l e^{-At}\,dt = -\frac{1}{A}[e^{-At}]_0^r = \frac{1}{A}(1-e^{-Ar}).$$

Hence the proof of the lemma is complete.

PROOF OF THE THEOREM : $f \in K_F(A, B) \Rightarrow F \in K(A, B)$. Now the proof follows from the above lemma and the definition of $F(z)$.

Let $P(A)$ denote the class of functions $p$ which are of the form $p(z) = 1/(1 + Aw(z))$, $0 < A \leqslant 1$, and $w$ analytic in $E$ with $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$. Let $K_\alpha(A)$ denote the class of functions $f$ which are of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in $E$ and satisfying the condition

$$\frac{D_z^{1+\alpha}\{z^\alpha f(z)\}}{(1+\alpha)D_z^\alpha\{z^{\alpha-1}f(z)\}} \in P(A),\ 0<\alpha<1$$

and $K_{-\alpha}(A)$ denote the class of functions $f$ analytic in $E$ and satisfying the condition

$$\frac{D_z^{1-\alpha}\{z^{-\alpha}f(z)\}}{(1-\alpha)D_z^{-\alpha}\{z^{-\alpha-1}f(z)\}} \in P(A),\ 0<\alpha<1.$$

Let $f*g(z)$ denote the Hadamard product of two functions $f(z)$ and $g(z)$, that is,

$$f*g(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho<1} f(\zeta)\, g(z/\zeta)\,\frac{1}{\zeta}\,d\zeta$$

and

$$D^\alpha f(z) = \left\{\frac{z}{(1-z)^{\alpha+1}}\right\} * f(z)\ (\alpha \geqslant -1).$$

Then the above relation implies

$$D^n f(z) = \frac{z\{z^{n-1}f(z)\}^{(n)}}{n!}$$

where $n = 0, 1, 2, \ldots$ :

Since $zD_z^{1+\alpha}\{z^\alpha f(z)\} = \Gamma(2+\alpha)D^{1+\alpha}f(z)$ and $zD_z^\alpha\{z^{\alpha-1}f(z)\} = \Gamma(1+\alpha)D^\alpha f(z)$ then

$$\frac{D_z^{1+\alpha}\{z^\alpha f(z)\}}{(1+\alpha)D_z^\alpha\{z^{\alpha-1}f(z)\}} = \frac{D^{\alpha+1}f(z)}{D^\alpha f(z)}.$$

Now we start by making the following remarks for the classes $K_\alpha(A)$ and $K_{-\alpha}(A)$ for $0<\alpha<1$ and $0<A\leqslant 1$.

*Remark* 1 : $f \in K_\alpha(A)$ if and only if $\frac{D^{\alpha+1}f(z)}{D^\alpha f(z)} \in P(A)$.

*Remark* 2 : $f \in K_{-\alpha}(A)$ if and only if $\frac{D^{1-\alpha}f(z)}{D^{-\alpha}f(z)} \in P(A)$.

*Theorem* 7—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E$ and satisfying the condition

$$\sum_{n=2}^{\infty} \frac{n}{2} [(n-1) + A(n+1)] \mid a_n \mid \leqslant A, \; 0 < A \leqslant 1 \qquad \ldots(15)$$

Then, for $0 < \alpha < 1$, $f \in K_\alpha(A)$.

PROOF : From (15) we obtain, using the fact $0 < \alpha < 1$,

$$\sum_{n=2}^{\infty} \frac{(n-1)\,\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+2)} \mid a_n \mid < \sum_{n=2}^{\infty} \frac{n(n-1)}{2} \mid a_n \mid$$

$$\leqslant A - A \sum_{n=2}^{\infty} \frac{n(n+1)}{2} \mid a_n \mid$$

$$\leqslant A - A \sum_{n=2}^{\infty} \frac{(n+\alpha)\,\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+2)} \mid a_n \mid .$$

Therefore for $z \in E$

$$\left| \sum_{n=2}^{\infty} \frac{(n-1)\,\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+2)} a_n z^{n-1} \right| \leqslant \sum_{n=2}^{\infty} \frac{(n-1)\,\Gamma(n+\alpha) \mid a_n \mid}{(n-1)!\,\Gamma(\alpha+2)} \mid z \mid^{n-1}$$

$$< A - A \sum_{n=2}^{\infty} \frac{(n+\alpha)\,\Gamma(n+\alpha) \mid a_n \mid}{(n-1)!\,\Gamma(\alpha+2)}$$

$$< A - A \left| \sum_{n=2}^{\infty} \frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)} a_n z^{n-1} \right|$$

$$\leqslant \left| A + A \sum_{n=2}^{\infty} \frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)} a_n z^{n-1} \right| . \qquad \ldots(16)$$

Define $w$ in $E$ by

$$w(z) = \frac{-\sum_{n=2}^{\infty} \frac{(n-1)\,\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+1)} a_n z^{n-1}}{A + A \sum_{n=2}^{\infty} \frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)} a_n z^{n-1}} = \frac{D^\alpha f(z) - D^{1+\alpha} f(z)}{A\, D^{1+\alpha} f(z)} .$$

Then $w(0) = 0$, $w$ is anylytic in $E$ and $| w(z) | \leqslant 1$, If $z \in E$, from (16).

Hence

$$\frac{D^{1+\alpha} f(z)}{D^{\alpha} f(z)} = \frac{1}{1 + Aw(z)}.$$

This shows that $f \in K_{\alpha}(A)$ is view of Remark 1.

*Remark* 3: Put $A = 1$ in Theorem 7. Then condition (15) reduces to $\sum_{n=2}^{\infty} n^2 | a_n | \leqslant 1$ which is less restrictive than Owa's condition $\sum_{n=2}^{\infty} n(n+2) | a_n | \leqslant 1$ in Theorem 5[3], though the conclusion is Re $\left\{ \frac{D^{1+\alpha} f(z)}{D^{\alpha} f(z)} \right\} > \frac{1}{2}$, the same as Owa's in his theorem.

*Corollary* 5—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E$ and satisfy the condition (15). Then, for $0 < \alpha < 1$, the Hadamard product $f * f(z)$ is in the class $K_{\alpha}(A)$.

PROOF: We have $f * f(z) = z + \sum_{n=2}^{\infty} a_n^2 z^n$. Since condition (15) implies that $| a_n | < 1$, we have

$$\left| \sum_{n=2}^{\infty} \frac{(n-1)\,\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+2)} a_n^2 z^{n-1} \right| \leqslant \sum_{n=2}^{\infty} \frac{(n-1)\,\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+2)} | a_n^2 | \, | z^{n-1}$$

$$< \sum_{n=2}^{\infty} \frac{(n-1)\,\Gamma(n+\alpha) | a_n | \leqslant z |^{n-1}}{(n-1)!\,\Gamma(\alpha+2)}$$

$$< A - A \sum_{n=2}^{\infty} \frac{(n+\alpha)\,\Gamma(n+\alpha)}{(n-1)!\,\Gamma(\alpha+2)} | a_n |$$

$$\leqslant \left| A + A \sum_{n=2}^{\infty} \frac{\Gamma(n+\alpha+1)}{(n-1)!\,\Gamma(\alpha+2)} a_n z^{n-1} \right|.$$

The proof is completed as in the case of Theorem 7.

*Theorem* 8—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E$ and satisfy the condition

$$\sum_{n=2}^{\infty} [n - 1 - An] | a_n | \leqslant A, \; 0 < A \leqslant 1. \qquad \ldots(17)$$

Then, for $0 < \alpha < 1$, $f \in K_{-\alpha}(A)$.

Proof is similar to that of Theorem 7.

*Corollary* 6—Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E$ and satisfy the condition (17). Then, for $0 < \alpha < 1$, the Hadamard product $f * f(z)$ is in the class $K_{-\alpha}(A)$.

## References

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