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SPLITTING OF RING EXTENSIONS

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If $R \subset S$ is an extension of rings making S into a projective right R-module then the inclusion splits if S is faithfully projective[2]. In this note we show that if R is commutative, the result is still true if S is R-projective.

All rings considered will have identities and a ring homomorphism will take identity to identity. If $R \subset S$ is an extension of rings making S into a projective right R-module then R is a direct summand of S iff S is faithfully projective. This is shown in Cartzen *et al.*[2] and the authors also give an example of a ring extension in which S is R-projective but not faithfully projective. The example involved non-commutative rings. In this note we shall show that such a situation cannot happen if R is commutative. This is shown in Theorem 1.

Theorem 1—Suppose R is a commutative ring and $R \subset S$, a ring extension, making S into a projective R-module. Then R is a direct summand of S as R-module.

This is mentioned as an exercise Bourbaki[1] [Ex 5.4, p. 176], when S is finitely generated.

Lemma—Suppose $R \subset S$ is a ring extension. Then S is faithfully flat over R if and only if S/R is R-flat.

The proof is given in Bourbaki[(1)].

PROOF OF THE THEOREM : Let $p \in \operatorname{Spec} R$; Localizing the exact sequence, $0 \to R \to S \to S/R \to 0$ at p, one obtains the exact sequence.

$0 \to R_p \to S_p \to (S/R)_p \to 0$ of R_p-modules. By a theorem of Kaplansky[4], the R_p-projective module S_p must be free and hence faithfully flat over R_p. By above Lemma, $(S/R)_p$ is R_p-flat. This being true for every $p \in \operatorname{Spec} R$, S/R must be R-flat. Hence by Lemma above S must be faithfully flat as R-module. To conclude R as a direct summand of S as R-module, one could make use of Theorem 1

of Cartzen *et al.*[2]; or one could use the criterion for the projectivity given in Jothilingam[3] to conclude that S/R is R-projective.

Corollary 1—Let R_0 be a commutative ring and $R_0 \subset R$ be an extension of rings. If there exists a ring S such that $R_0 \subset R \subset S$ with S R_0-projective, then R_0 is a direct summand of R as R_0-module.

PROOF : By Theorem 1 there exists a splitting $S \to R_0$ as R_0-module. Restrict this to R to get a splitting $R \to R_0$.

Corollary 2—Let $R_0 \subset R \subset S$ be as above with S projective as R_0-module, then R is a direct summand of $S \oplus R/R_0$ as R_0-modules.

PROOF : Consider a decomposition of S : $S = R_0 \oplus S_1$ as R_0-modules. Then $R = R_0 \oplus R \cap S_1$. We obtain a monomorphism f of R into $S \oplus R/R_0 = R_0 \oplus R/R_0 \oplus S_1$ by setting $f(x + y) = x + (y + R_0) + y$, where $x \in R_0$, $y \in R \cap S_1$. It is clear that the mapping $g : R/R_0 \to R \cap S_1$ defined by $g(y + R_0) = y$ is an isomorphism. On the other hand, $S_1 \oplus R/R_0 = S_1 \oplus R/R_0(g)$, where $R/R_0(g) = \{y + g(y)/y \in R/R_0\}$. Hence $R \cong R_0 \oplus R/R_0(g)$ and $R_0 \oplus R/R_0(g) \oplus S_1 = R_0 \oplus R/R_0 \oplus S_1 = S \oplus R/R_0$.

Hence R is a direct summand of $S \oplus R/R_0$ as R_0-module.

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