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ON THE LATTICE OF ONE-POINT NEAR-COMPACTIFICATIONS

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In this paper we study the lattice of one-point near-compactifications and show that there exists a projective maximum in the set of all one-point near-compactifications of a locally nearly compact space. We also demonstrate the utlitiy of these extensions by providing an application.

1. INTRODUCTION AND PRELIMINARIES

A topological space is said to be nearly compact[1,2] if and only if it is Hausdorff and every open cover $\mathscr{U}$ of X admits a finite subfamily $\mathscr{U}'$ ($\subset \mathscr{U}$) such that $X = \cup \{\text{int cl } U \mid U \in \mathscr{U}'\}$ (or equivalently, it is Hausdorff, every regular open cover of X admits a finite subcover of X). Carnahan[3] introduced the concept of N-closed sets and investigated the class of locally nearly compact spaces. Noiri[4-7] continued these investigations to describe in his papers further properties of locally nearly compact spaces. Hermann[8] describes two one-point near-compactifications of locally nearly compact spaces. Indeed Hermann shows that if (X, τ) is locally nearly compact then there exist one-point near-compactifications (Y,τ^*) and (Y, τ^v) such that $(Y, \tau^\wedge)$ is a projective minimum in a large class of extensions of X (i.e. those extensions of X in which X is open) and (Y, τ^*) is almost a projective maximum in the class of all one-point near-compactifications. Earlier Obreanu[9] described the one-point H-closed extensions of locally H-closed spaces. Porter[10] continued the study of locally H-closed spaces to great depths.

In this paper we describe an one-point near-compactification $(X^*, \tau^\#)$ of a locally nearly compact space (X, τ) such that $(X^*, \tau^\#)$ is projective maximum in the class of all one-point near-compactifications of (X, τ). We apply these results to give a different proof of a well-known result of Vermeer[11] that the projective objects in the category of nearly compact spaces and continuous maps are precisely finite spaces. Further we show that if $\mathscr{K}$ is the set of all Hausdorff near-compactifications of a Hausdorff space X, then $\mathscr{K}$ has a projective minimum if X is locally nearly compact. We also indicate the relationship in the class of all Hausdorff almost regular spaces between the one-point U-closed extensions of a locally U-closed spaec and the one-point near-compactifications of a locally nearly compact space.

Throughout this paper all spaces are assumed to be Hausdorff unless indicated to the contrary. Further we shall use the following notations. Let (X, τ) be a topological space and $A \subset X$. $\text{Int}_\tau A$ and $\text{Cl}_\tau A$ denote the interior and closure of A in (X, τ). (When the context of the space and its topology are clear we suppress the topology involved). The only exception to this convention occurs in the beginning of section 3 of this paper, where $\text{cl}_\theta\ U$ represents the θ-closure of U introduced by Velicko[12]. τ/A denotes the relative topology of τ on A. We write nbd for neighbourhood.

A subset A $(\subset X)$ is an N-set if and only if every open cover $\mathscr{U}$ of A in X has a finite subfamily $\mathscr{U}'$ $(\subset \mathscr{U})$ such that $A \subset \cup \{\text{int cl } U \mid U \in \mathscr{U}'\}$ (or equivalently, every cover of A by sets regular open in X has a finite subcover). Carnahan[3] uses the terminology N-closed relative to τ or simply N-closed for an N-set. A space (X, τ) is called locally nearly compact if and only if for each $x \in X$ there exists an open nbd U of x such that cl U is an N-set. It is well known that X is nearly compact if and only if X_s (the semiregularization of X) is compact or equivalently, X is H-closed and Urysohn or X is H-closed and almost regular. Further if X is locally nearly compact, then X is almost regular.

If (Y, σ) and (X, τ) are two spaces such that $X \subset Y$, Y is called an one-point near-compactification of X if and only if $\sigma/X = \tau$, $\text{cl}_\sigma\ (X) = Y$, $Y - X$ is a singleton and Y is nearly compact. If $\mathscr{L}(X)$ is a set of extensions of X and Y, $Z \in \mathscr{L}(X)$, then Y is called projectively larger (smaller) than Z if there exists a continuous surjection $f: Y \to Z$ $(f: Z \to Y)$ such that $f(x) = x$ for all $x \in X$. An element $Y \in \mathscr{L}(X)$ is called a projective maximum (minimum) if and only if for each $Z \in \mathscr{L}(X)$, Y is projectively larger (smaller) than Z.

2. One-point Extensions

In this section we study the lattice of one-point near-compactifications. In his paper Hermann[8] describes two one-point near-compactifications of a locally nearly compact space (X, τ). These extensions are $(X^*, \tau^\wedge)$ and (X^*, τ^*) where $X^* = X \cup \{p\}$ and $p \notin X$. One may refer to Noiri[7] (p. 183) for the definitions of $\tau^\wedge$ and τ^* on X^*.

Theorem 2.1 (Hermann[8])— Let (X, τ) be a locally nearly compact space, but not H-closed. Then

(i) $(X^*, \tau^\wedge)$ and (X^*, τ^*) are nearly compact Hausdorff spaces

(ii) $(X^*, \tau^\wedge)$ is a projective minimum in the class of all near-compactifications of (X, τ) in which X is open (in particular, in the class of all one-point near-compactifications of (X, τ))

(iii) if (Y, σ) is any one-point near-compactification then there exists a bijection $f: (X^*, \tau^*) \to (Y, \sigma)$ which is continuous and fixed on X and almost continuous on $X^* - X$. □

Let us define another one-point near-compactification which clarifies the lattice of one-point near-compactifications of (X, τ). Let $\alpha = \{U \mid U \in \tau$ and $X-$int cl U is an N-set$\}$. α is an open filter base; indeed an open filter. For, if V is open and $V \supset U \in \alpha$, then X-int cl $V \subset X-$int cl U so that $X-$int cl V is an N-set and hence $V \in \alpha$. Define $\tau^* = \tau \cup \{\{p\} \cup U \mid U \in \alpha\}$

Theorem 2.2—Let (X, τ) be a locally nearly compact space, but not H-closed. Then

(i) $(X^*, \tau^{\#})$ is a nearly compact Hausdorff space

(ii) $(X^*, \tau^{\#})$ is a projective maximum in the class of all one-point near-compactifications of (X, τ).

(iii) In general, $\tau^* \subset \tau^{\#}$.

PROOF: That $(X^*, \tau^{\#})$ is Hausdorff follows from (X, τ) is Hausdorff and locally nearly compact.

Let $\mathscr{W}$ be a $\tau^{\#}$ regular open cover of X^*. Then there exists a τ-open set V_0 such that $\{p\} \cup V_0 \subset W_0$ and $X-$int cl V_0 is an N-set and $W_0 \in \mathscr{W}$. $W_0 \cap X$ is regular open in X and hence int cl $V_0 \subset \mathscr{W}_0 \cap X \subset W_0$. Let $\mathscr{W}' = \{W \cap X \mid W \in \mathscr{W}\}$. $\mathscr{W}'$ is a regular open cover of $X-$int cl V_0 so that there exists $W_i \in \mathscr{W}$ $(i = 1, 2 \ldots\ldots n)$ such that $X-$int cl $V_0 \subset \cup (W \cap X)$. Hence $X^* = \bigcup_{i=0}^{n} W_i$. Thus $(X^*, \tau^{\#})$ is nearly compact.

Let (Y, σ) be any one-point near-compactification of (X, τ). Let $Y = X \cup \{\eta\}$. Define $f: X^* \to Y$ as follows: $f(x) = x$ for all $x \in X$ and $f(p) = \eta$. Clearly it is enough to verify the continuity of f at p. Let U be a σ-open nbd of η. $Y-\text{int}_\sigma \text{cl}_\sigma U$ is regular closed in (Y, σ) and hence an N-set in Y so that $Y-\text{int}_\sigma \text{cl}_\sigma U$ is an N-set in X. Since $X-$int cl $(U \cap X) = Y-\text{int}_\sigma \text{cl}_\sigma U$, $(U \cap X) \cup \{p\} = f^{-1}(U)$ is a basic open nbd of p in $(X^*, \tau^{\#})$.

That $\tau^* \subset \tau^{\#}$ is clear. □

Proposition 2.3—Let (X, τ) be a locally nearly compact space. Let (X^*, σ) be any one-point near-compactification of (X, τ). If $X^* = X \cup \{p\}$ and τ^{Δ} and $\tau^{\#}$ are as in Theorems 2.1 and 2.2 respectively, then $\tau^{\Delta} \subset \sigma \subset \tau^{\#}$.

PROOF : In the last part of the proof of Theorem 2.2, if we take $\eta = p$, f becomes the identity function which is continuous so that $\sigma \subset \tau^{\#}$. Similarly $\tau^{\wedge} \subset \sigma$. □

The following result is a sort of a converse of the Proposition 2.3

Theorem 2.4—Let (X, τ) be locally nearly compact. Let $X^* = X \cup \{p\}$. Let σ_1 and σ_2 be two Hausdorff topologies on X^* such that (i) $\sigma_1/X = \sigma_2/X = \tau$ (ii) $\sigma_1 \subset \sigma_2$ and (X^*, σ_2) is nearly compact. Then (X^*, σ_1) is nearly compact.

PROOF : If $\mathscr{U}$ is a regular open cover (regular open in σ_1) of X^* and U a regular open (in σ_1) nbd of p with $U \in \mathscr{U}$, then $U \cap X$ is regular open (in τ). Note that $X - U \cap X (= X^* - U)$ is τ-regular closed and also σ_2-closed. Thus $X - U \cap X = \mathrm{cl}_\tau \mathrm{int}_\tau (X - U \cap X) = \mathrm{cl}_\tau \mathrm{int}_{\sigma_2} (X - U \cap X) = (\mathrm{cl}_{\sigma_2} \mathrm{int}_{\sigma_2} (X - U \cap X)) \cap X$. Indeed $p \notin \mathrm{cl}_{\sigma_2} (X - U \cap X)$ so that $X - U \cap X = \mathrm{cl}_{\sigma_2} \mathrm{int}_{\sigma_2} (X - U \cap X)$. Hence $P = X - U \cap X = X^* - U$ is an N-set in (X^*, σ_2) so that $(P, (\sigma_2)_s/P)$ is compact. Since $p \notin P$, $(\sigma_2)_s/P = (\sigma_1)_s/P = \sigma_s/P$. Hence $(P, (\sigma_1)_s/P)$ is compact so that P is an N-set in (X^*, σ_1). Since $X^* = U \cup P$ (where $U \in \mathscr{U}$), (X^*, σ_1) is nearly compact. □

It is well-known that if U is regular open in Y and X a dense subspace of Y, then $U \cap X$ is regular open in X. Indeed, if $X \subset Y$ and X is dense in Y then $(\sigma/X)_s = \sigma_s/X$ where σ is the topology on Y. Hence if (Y, σ) is a Hausdorff extension of (X, τ), then (Y, σ_s) is a Hausdorff extension of (X, τ_s). It is also known that (X, τ) is locally nearly compact if and only if (X, τ_s) is locally compact. Thus

Proposition 2.5—If (Y, σ) is a Hausdorff extension of a locally nearly compact space (X, τ), then X is open in Y. □

As a consequence of Proposition 2.5, we get the following result which improves the result of Hermann quoted in 2.1 (ii) above.

Theorem 2.6—Let (X, τ) be a locally nearly compact space. Let $\mathscr{K}$ be the class of all Hausdorff extensions of (X, τ) such that $Y - X \neq \phi$ for each $Y \in \mathscr{K}$. Then (X^*, τ^*) is a projective minimum in $\mathscr{K}$. □

3. RELATION WITH U-CLOSED SPACES

An Urysohn space X is Urysohn-closed or U-closed if X is closed in every Urysohn space into which it can be embedded. An open cover $\mathscr{U}$ of a space X is called Urysohn cover if there exists an open cover $\mathscr{V}$ such that for each $V \in \mathscr{V}$ there exists $U \in \mathscr{U}$ satisfying $\mathrm{cl}\, V \subset U$. It is well-known that a Urysohn space X

is U-closed if and only if for every Urysohn cover $\mathscr{U}$ there exist $U_i \in \mathscr{U}$ ($i = 1, 2, \ldots\ldots n$) such that $X = \bigcup_{i=1}^{n} \text{cl } U_i$. The concepts of θ-closure of a subset and θ-closed subsets in a topological space was introduced by Velicko[12]. One can refer to Dickman and Porter[13] and Raghavan[14] for further details. It is also known that if U is an open subset of X, $\text{cl}_\theta\ U = \text{cl } U$. However $\text{cl}_\theta\ U$ need not be θ-closed. Let us now define a condition (L) as follows :

(L) : $\text{cl}_\theta\ \text{cl}_\theta\ U = \text{cl}_\theta\ U$ for every open $U \subset X$.

If X satisfies (L), then X is Urysohn; and if X is almost regular, X satisfies (L) Indeed if X is H-closed then the following are equivalent, namely: (i) X is Urysohn (ii) X satisfies (L) (iii) X is almost regular.

Let (X, τ) be a Urysohn space and $A \subset X$, $\mathscr{U}$ is called a Urysohn cover of A if $A \subset \cup \mathscr{U}$ and there exists an open cover $\mathscr{V}$ of A in X such that for each $V \in \mathscr{V}$ there exists $U \in \mathscr{U}$ with $\text{cl } V \subset U$. A is called U-closed relative to X or a U-set if for every Urysohn cover $\mathscr{U}$ of A, there exists $U_i \in \mathscr{U}$ ($i = 1, 2 \ldots\ldots n$) such that $A \subset \bigcup_{i=1}^{n} \text{cl } U_i$. A is called a U-closed set if $(A, \tau/A)$ is U-closed. X is called locally U-closed if each point X has an open nbd whose closure is a U-set.

The verification of the following proposition is left to the reader.

Proposition 3.1—(i) If X is U-closed and A is a θ-closed subset of X, then A is a U-set in X (ii) If X is a U-closed space satisfying (L) and A is regular closed subset of X, then A is a U-set in X. (iii) If X is Urysohn, $A \subset X$ and A is a U-set in X, then A is colsed. (iv) If X is Hausdorff satisfying (L) $A \subset X$ and A is a U-set in X, then A is θ-closed. □

In what follows HAR denotes the class of all Hausdorff almost regular spaces.

Theorem 3.2—Let $X \in$ HAR and $A \subset X$. Then the following are equivalent :

(i) A is an H-set

(ii) A is an N-set

(iii) A is a U-set.

PROOF : That (i) $\Rightarrow$ (ii) is proved in (Lemma 4; Noiri[6]). Also that (ii) $\Rightarrow$ (iii) is obvious. Now let as prove (iii) $\Rightarrow$ (i). Let $\mathscr{U}$ be an open cover of A in X. Let $U \in \mathscr{U}$ and $X \in U$. Since X is almost regular, there exists an open nbd V_x of x such that $x \in V_x \subset \text{cl } V_x \subset \text{int cl } U$. $\mathscr{W} = \{\text{int cl } U \mid U \in \mathscr{U}\}$ and $\mathscr{V} = \{V_x \mid x \in A\}$ are open covers of A such that for each $V \in \mathscr{V}$ there exists $W \in \mathscr{W}$ such that $\text{cl } V \subset W$. Thus $\mathscr{W}$ is Urysohn and hence there exist $U_i \in \mathscr{U}$ ($i = 1, 2, \ldots\ldots n$) such that $A \subset \bigcup_{i=1}^{n} \text{cl int cl } U_i = \bigcup_{i=1}^{n} \text{cl } U_i$. Thus A is an H-set. □

Theorem 3.3—Let $X \in$ HAR. Then the following are equivalent ;

(i) X is (locally) H-closed

(ii) X is (locally) nearly compact

(iii) X is (locally) U-closed. □

Thus Theorems 2.1 and 2.2 attain the following status in the class HAR.

Theorem 3.4—Let $(X, \tau) \in$ HAR and be a locally U-closed space which is not U-closed. Then X admits two one-point U-closed extensions in HAR namely $(X^*, \tau^{\wedge})$ and $(X^*, \tau^{\#})$ such that

(i) $X^* = X \cup \{p\}$ where $p \notin X$

(ii) $\tau^{\wedge} = \tau \cup \{p\} \cup V \mid V \in \beta\}$ where β is the open filter generated by $\{U \mid U \in \tau$ and $X - U$ is a U-set$\}$

(iii) $\tau^{\#} = \tau \cup \{\{p\} \cup V \mid X$-int cl V is a U-set$\}$

(iv) $(X^*, \tau^{\wedge})$ is a projective minimum in the set of all one-point U-closed extensions in HAR of (X, τ).

(v) $(X^*, \tau^{\#})$ is a projective maximum in the set of all one-point U-closed extensions in HAR of (X, τ). □

The following brings out the significance of Theorem 2.4.

Theorem 3.5—Let X be a locally nearly-compact space. Let $\mathcal{L}(X)$ denote the set all one-point near-compactifications of X. Let $\mathcal{L}'(X)$ denote the set of all one-point H-closed extensions of X treated as a locally H-closed space. Then $\mathcal{L}(X) = \mathcal{L}'(X)$ (where the same point p serves as the extra point in every one-point extension of X).

PROOF : The projective maximum and projective minimum im $\mathcal{L}'(X)$ are found in (Theorems 2.2, 2.3 and 2.4; Porter[10]). In the Theorem 2.2 of Porter[10], to describe the minimum one may as well take the basic nbds of p such that their complements in X^* are H-sets in X. Similar observation applies to Theorems 2.3 and 2.4 of Porter[10] where he describes the projective maximum. Since X is almost regular, by virtue of Theorem 3.2, the basic nbds of p in these two topologies (the minimum and maximum in $\mathcal{L}'(X)$) are precisely those described in Theorems 2.1 (ii) and 2.2 respectively. Thus the maximum and minimum of $\mathcal{L}(X)$ and $\mathcal{L}'(X)$ agree. Moreover $\mathcal{L}(X) \subset \mathcal{L}'(X)$. If (X^*, ρ) is any one-point H-closed extension (not necessarily almost regular), then $\rho \subset \tau^{\#}$, so that by Theorem 2.4, (X^*, ρ) is nearly compact. □

4. An Application

Now we give an alternate proof of a result of Vermeer[11] mentioned earlier in which we use Theorems 2.2 and 2.4. For the definition of projective objects in a category and other related details, one may refer to either Herrlich[15] or Raghavan and Reilly[15].

Theorem 4.1 (Vermeer[11])—In the category of nearly compact spaces and continuous maps X is projective if and only if X is finite.

Proof : If X is finite then clearly X is projective. Conversely, let (X, τ) be projective. Suppose X is not finite so that X is not discrete. Then there exists an element a in X which is not isolated. Let $Y = X - \{a\}$ and $\tau/Y = \sigma_1$. Let N be the set of non-zero positive integers with discrete topology and (N^*, σ_2) be its one-point compactification where $N^* = N \cup \{\omega\}$.

Let $B = Y \times N^*$, and $A = B \cup \{p\}$ where $p \notin Y \times N^*$. Define a topology ρ on A such that $\rho/(Y \times N^*) = \sigma_1 \times \sigma_2$ $(= \sigma$ say) and the basic ρ-open nbds of p are of the form $(Q \times N) \cup \{p\}$ where Q is a deleted open nbd of a. Thus we obtain a Hausdorff space (A, ρ).

Y is locally nearly compact (by Theorem 3.3; Hermann[8]). Since N^* is nearly compact, (B, σ) is locally nearly compact (by Theorem 2.2; Herrington[17]). Further B is ρ-open in A, $\mathrm{cl}_\rho\ B = A$ and A is an one-point extension of B. Since (X,τ) is an one-point near-compactification of (Y, σ_1), if $Q \cup \{a\} \in \tau$, then $Q \cup \{a\} \in \sigma_1^*$ taking $Y^* = Y \cup \{a\} = X$ (by Proposition 2.3) so that $B - \mathrm{int}_\sigma\ \mathrm{cl}_\sigma (Q \times N) = (Y - \mathrm{int}_{\sigma_1}\ \mathrm{cl}_{\sigma_1}\ Q) \times N^*$ is an N-set in (B, σ). Hence $(Q \times N) \cup \{p\} \in \sigma^*$ taking $B^* = B \cup \{p\} = A$. Thus $\rho \subset \sigma^*$. Since the conditions of Theorem 2.4 are satisfied, (A, ρ) is nearly compact.

Now if we define a new topology ρ' on $A = (Y \times N^*) \cup \{p\}$ by letting $Y \times N^*$ have the product topology and taking the basic ρ'-open nbds of ρ to be of the form $(Q \times N^*) \cup \{p\}$ where Q is an open deleted nbd of a, then by argument similar to that in the previous paragraph (A, ρ') is nearly compact.

The remainder of the proof consists of obtaining a violation of continuity similar to that in the proof of Theorem 2.1 of Raghavan and Reilly'[6]. □

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