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ON THE LOGARITHMIC ORDER OF ENTIRE DIRICHLET SERIES

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In this paper, we obtain some formulae in terms of the coefficients a_n, for logarithmic order and lower logarithmic order, for an entire Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$, $(s = \sigma + it,\ \lambda_{n+1} > \lambda_n,\ \lambda_1 \geqslant 0,\ \lambda_n \to \infty$ with n, $\limsup_{n\to\infty} (\log n/\lambda_n = 0)$, constructed with the help of two other series.

Theorem 2 of this paper is an extension of a similar result due to Jain (1971) and whose Taylor series analogue is due to Kamthan (1966).

§1. Let $f(s)$ be an entire function represented by the Direchlet reries[1]

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n),\ s = \sigma + it$$

where the constants λ_n satisfy the following conditions :

$$0 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots;\ \lambda_n \to \infty \text{ as } n \to \infty \qquad \ldots(1.1)$$

and

$$\lim_{n\to\infty} \sup \frac{\log n}{\lambda_n} = 0. \qquad \ldots(1.2)$$

The Ritt order ρ $(0 \leqslant \rho \leqslant \infty)$ of $f(s)$ is defined by[2]

$$\lim_{\sigma\to\infty} \sup \frac{\log\log M(\sigma)}{\sigma} = \rho \qquad \ldots(1.3)$$

where $M(\sigma) = \underset{-\infty<t<\infty}{l.u.b.} |f(\sigma + it)|$. For the class of entire functions of Ritt order zero, the logarithmic order H and lower logarithmic order h have been defined by Reddy[3]; for an analogous definition and similar results on logarithmic orders in Taylor series, see Kamthan[4]. Thus

$$\lim_{\sigma\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\log\sigma} = \begin{matrix}H \\ h\end{matrix},\ 1 \leqslant h \leqslant H \leqslant \infty. \qquad \ldots(1.4)$$

It is also known that (Reddy[3])

$$\limsup_{n\to\infty} \frac{\log \lambda_n}{\log (\lambda_n^{-1} \log |a_n|^{-1})} = H - 1 \qquad \ldots(1.5)$$

and if $\log \lambda_n \sim \log \lambda_{n+1}$, then

$$\liminf_{n\to\infty} \frac{\log \lambda_n}{\log (\lambda_n^{-1} \log |a_n|^{-1})} \leqslant h - 1. \qquad \ldots(1.6)$$

Also, we have that if $(\log |a_n/a_{n+1}|)/(\lambda_{n+1} - \lambda_n)$ is a non-decreasing function of n for $n > n_0$, then (Reddy[3])

$$\liminf_{n\to\infty} \frac{\log \lambda_n}{\log (\lambda_n^{-1} \log |a_n|^{-1})} \geqslant h - 1. \qquad \ldots(1.7)$$

We say that $f(s)$ is of regular logarithmic growth if $1 < h = H < \infty$.

In the present paper we investigate various relations between logarithmic order and lower logarithmic order of two or more entire Dirichlet series.

§2. Now we prove a result whose Taylor series analogue is already known (cf. Kamthan[5]).

Theorem 1—Let $f_i(s) = \sum_{n=1}^{\infty} a_{i,n} \exp(s\lambda_{i,n})$, $(i = 1, 2)$, be two entire functions of logarithmic orders H_i $(1 < H_i < \infty)$ and lower logarithmic orders h_i $(1 < h_i < \infty)$. Suppose that there exist sequences $\{\lambda_n\}$ and $\{a_n\}$ such that

(i) $\log \lambda_n \sim \log \lambda_{1,n} \sim \log \lambda_{2,n}$,

(ii) $\log \lambda_n \sim \log \lambda_{n+1}$,

(iii) $\log |a_n|^{-1} \sim (\log |a_{1,n}|^{-1})^{\alpha} (\log |a_{2,n}|^{-1})^{1-\alpha}$, $0 < \alpha < 1$

(iv) $\dfrac{\log |a_{i,n}/a_{i,n+1}|}{\lambda_{i,n+1} - \lambda_{i,n}}$ and $\dfrac{\log |a_n/a_{n+1}|}{\lambda_{n+1} - \lambda_n}$

from non-decreasing functions of n for all sufficiently large n. Then

$$f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$$

is an entire function of logarithmic order H $(1 < H < \infty)$ and lower logarithmic order h $(1 < h < \infty)$ such that

$$\frac{1}{h-1} > \frac{\alpha}{H_1 - 1} + \frac{1-\alpha}{h_2 - 1} > \frac{1}{H-1} \qquad \ldots(2.1)$$

and

$$\frac{1}{h-1} \geqslant \frac{\alpha}{h_1-1} + \frac{1-\alpha}{H_2-1} \geqslant \frac{1}{H-1}. \qquad \ldots(2.2)$$

PROOF : Using (i), (ii), (iii), Theorems 1 and 3 of Reddy[3], we get

$$\limsup_{n\to\infty} \frac{\log\ (\lambda_n^{-1} \log \mid a_n \mid^{-1})}{\log \lambda_n}$$

$$\geqslant \liminf_{n\to\infty} \frac{\alpha \log\ (\lambda_{1,n}^{-1} \log \mid a_{1,n} \mid^{-1})}{\log \lambda_{1,n}}$$

$$+ \limsup_{n\to\infty} \frac{(1-\alpha) \log\ (\lambda_{2,n}^{-1} \log \mid a_{2,n} \mid^{-1})}{\log \lambda_{2,n}}$$

$$\geqslant \liminf_{n\to\infty} \frac{\log\ (\lambda_n^{-1} \log \mid a_n \mid^{-1})}{\log \lambda_n}.$$

Hence

$$\frac{1}{h-1} \geqslant \frac{\alpha}{H_1-1} + \frac{1-\alpha}{h_2-1} \geqslant \frac{1}{H-1}.$$

Similarly, we can prove that

$$\frac{1}{h-1} \geqslant \frac{\alpha}{h_1-1} + \frac{1-\alpha}{H_2-1} \geqslant \frac{1}{H-1}.$$

This completes the proof of Theorem 1.

Also, it can easily be shown that

$$\frac{1}{H-1} > \frac{\alpha}{H_1-1} + \frac{1-\alpha}{H_2-1}. \qquad \ldots(2.3)$$

Combining the results (2.1) and (2.2) with (2.3), we obtain

$$\frac{\alpha}{h_1-1} + \frac{1-\alpha}{H_2-1} > \frac{1}{H-1} \geqslant \frac{\alpha}{H_1-1} + \frac{1-\alpha}{H_2-1} \qquad \ldots(2.4)$$

$$\frac{\alpha}{H_1-1} + \frac{1-\alpha}{h_2-1} \geqslant \frac{1}{H-1} \geqslant \frac{\alpha}{H_1-1} + \frac{1-\alpha}{H_2-1} \qquad \ldots(2.5)$$

and these inequalities lead to

Theorem 2—If all the conditions are same as in Theorem 1, and either of the two functions $f_1(s)$ or $f_2(s)$ is of regular logarithmic growth, then

$$\frac{1}{H-1} = \frac{\alpha}{H_1-1} + \frac{1-\alpha}{H_2-1}. \qquad \ldots(2.6)$$

Remark: This theorem generalizes a result of Jain[6] for $\alpha = 1/2$, which he obtained under more restrictive condition that both $f_1(s)$ and $f_2(s)$ are of regular logarithmic growth. (2.3) is also the generalization of an inequality due to Jain[6].

Theorem 3—If all the conditions are same as in Theorem 1 except (iii), which may be read as

(iii)' $\log(\lambda_n^{-1}(\log|a_n|^{-1}) \sim \sqrt{\log(\lambda_n^{-1}\log|a_{1,n}|^{-1})^{\alpha}\log(\lambda_n^{-1}\log|a_{2,n}|^{-1})^{\beta}}$

$(0 < \alpha < \infty, 0 < \beta < \infty)$ then $f(s)$ is an entire function and

$$\sqrt{\frac{\alpha\beta}{(h_1-1)(h_2-1)}} > \frac{1}{h-1} \geqslant \sqrt{\frac{\alpha\beta}{(H_1-1)(h_2-1)}} \geqslant \frac{1}{H-1}$$

$$> \sqrt{\frac{\alpha\beta}{(H_1-1)(H_2-1)}} \qquad \ldots(2.7)$$

$$\sqrt{\frac{\alpha\beta}{(h_1-1)(h_2-1)}} \geqslant \frac{1}{h-1} > \sqrt{\frac{\alpha\beta}{(h_1-1)H_2-1)}} > \frac{1}{H-1}$$

$$\geqslant \sqrt{\frac{\alpha\beta}{(H_1-1)(H_2-1)}}. \qquad \ldots(2.8)$$

PROOF: Since $f_i(s)$ are entire functions, therefore, for a fixed large number n_0,

$$((\log|a_{i,n}|^{-1})/\lambda_{i,n}) > k, n > n_0,$$

where k is positive and arbitrarily large. Using (i) and (iii)', we get, for sufficiently large n,

$$\frac{\log|a_n|^{-1}}{\lambda_n} > K, K = k^{(\alpha\beta)1/2}.$$

Therefore $f(s)$ is an entire function.

Now, we have

$$\lim_{n\to\infty}\inf\frac{\log(\lambda_n^{-1}|a_{i,n}|^{-1})}{\log\lambda_n} = \frac{1}{H_1-1}.$$

This gives, for any $\epsilon > 0$ and $n > n_0$,

$$\left\{\frac{\log\,(\lambda_n^{-1}\log\mid a_n\mid^{-1})}{\log\lambda_n}\right\}^2 > \alpha\beta\left\{\frac{1}{H_1-1}-\epsilon\right\}\left\{\frac{1}{H_2-1}-\epsilon\right\}.$$

Thus,

$$\frac{1}{H-1} > \sqrt{\frac{\alpha\beta}{(H_1-1)(H_2-1)}}. \qquad \ldots(2.9)$$

Again, we have, $\log\lambda_{l,n} \sim \log\lambda_{l,n+1}$ in view of (i) and (ii). Hence, if (iv) also hold, then

$$\lim_{n\to\infty}\sup\frac{\log\,(\lambda_n^{-1}\log\mid a_{l,n}\mid^{-1})}{\log\lambda_n} = \frac{1}{h_l-1}$$

for $f_l(s)$. Now, proceeding as before, we obtain

$$\frac{1}{h-1} \leqslant \sqrt{\frac{\alpha\beta}{(h_1-1)(h_2-1)}}. \qquad \ldots(2.10)$$

Further, we have

$$\lim_{n\to\infty}\sup\frac{\log\,(\lambda_n^{-1}\log\mid a_n\mid^{-1})}{\log\lambda_n} \geqslant \left\{\lim_{n\to\infty}\inf\sqrt{\frac{\log\,(\lambda_{1,n}^{-1}\log\mid a_{1,n}\mid^{-1})^{\alpha}}{\log\lambda_{1,n}}}\right\}$$

$$\left\{\lim_{n\to\infty}\sup\sqrt{\frac{\log\,(\lambda_{2,n}^{-1}\log\mid a_{2,n}\mid^{-1})^{\beta}}{\log\lambda_{2,n}}}\right\}$$

$$= \lim_{n\to\infty}\inf\frac{\log\,(\lambda_n^{-1}\log\mid a_n\mid^{-1})}{\log\lambda_n}.$$

Thus,

$$\frac{1}{h-1} > \sqrt{\frac{\alpha\beta}{(H_1-1)(h_2-1)}} > \frac{1}{H-1}. \qquad \ldots(2.11)$$

Similarly, we can prove that

$$\frac{1}{h-1} \geqslant \sqrt{\frac{\alpha\beta}{(h_1-1)(H_2-1)}} \geqslant \frac{1}{H-1}. \qquad \ldots(2.12)$$

Combining the inequalities (2.9) to (2.12), we get the desired result.

From (2.7) and (2.8), we have

Theorem 4—If all the conditions are the same as in Theorem 3, then

$$\frac{1}{H-1} = \sqrt{\frac{\alpha\beta}{(H_1-1)(H_2-1)}}$$

provided either of the two functions $f_1(s)$ or $f_2(s)$ is of regular logarithmic growth.

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