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ON WEIGHTED L^p-CONVERGENCE OF CERTAIN LAGRANGE INTERPOLATION*

K. B. SRIVASTAVA

Department of Mathematics, I. I. T., Kanpur 208016

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The present paper is a modest contribution to the theory of weighted mean convergence and mean convergence of the Lagrange interpolation built on the set of nodes consisting of the zeros of $\pi_{n+1}(x) = (1-x^2)\ P_n'(x)$, $P_n(x)$ being the nth Legendre polynomial with normalization $P_n(1) = 1$.

1. PRELIMINARIES AND RESULTS

In this paper we intend to investigate the mean and weighted mean convergence of the Lagrange interpolation polynomial $F_n(f, x, \pi) \equiv F_n(f, \pi)$ constructed on the zeros of $\pi_{n+1}(x) = (1-x^2)\ P_n'(x)$, $P_n(x)$ being the nth Legendre polynomial with normalization $P_n(1) = 1$. Let us denote by $\phi_{kn}(x) \equiv \phi_k(x)$, $K = \overline{O, n}$; the fundamental polynomial of Lagrange interpolation for the corresponding node system, then

$$F_n(f, \pi) = \sum_{k=0}^{n} f(x_k)\ \phi_k(x) \qquad \ldots(1.1)$$

$$\phi_0(x) = \frac{(1+x)\ P_n'(x)}{n(n+1)} = (-1)^{n-1}\ \phi_n(-x)$$

$$\phi_k(x) = \frac{(1-x^2)\ P_n'(x)}{n(n+1)\ P_n(x_k)\ (x_k - x)}, \quad k = \overline{1, n-1}. \qquad \ldots(1.2)$$

We state the following main.

Theorem 1.1 — Let $f \in C$ and $F_n(f, \pi)$ be defined by (1.1) and (1.2), then, we have

$$\lim_{n\to\infty} \int_{-1}^{1} |\ F_n(f, \pi) - f\ |^p\ dx = 0 \qquad \ldots(1.3)$$

provided

$$\lim_{n\to\infty} n^{1/2}\ \omega_f\left(\frac{1}{n}\right) = 0 \qquad \ldots(1.4)$$

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for $p > 3$. Here $\omega_f(\cdot)$ stands for the modulus of continuity of f.

Theorem 1.2—Let $\rho > 1$ be any positive integer, then with the conditions of Theorem 1.1, we have

$$\lim_{n\to\infty} \int_{-1}^{1} [F_n(f, \pi) - f]^{2\rho} (1-x^2)^{(\rho-1)/2}\, dx = 0. \qquad ...(1.5)$$

Remark 1 : We can replace the exponent in Theorem 1.2 by any $p > 0$ if the corresponding change in weight function is also made by making use of Minkovski-Holder's inequality for integrals.

Remark 2 : It is not accessible to take Jacobi abscissas for changing weight functions. We may not be able to obtain better results for weighted mean convergence as evidenced by Askey (1972) and Feldheim (1937).

Mention may be made of a few results of this type established by Erdös and Feldheim[5] and Verma and Vertesi[12]. For general setting one is referred to Askey[1,2]. The background history of such results can be found in Erdös and Turán[3,4], Féjer[6] Nevai[9], Szegö[11], Rivlin[10] and Vertesi[13].

2. Proofs of Theorems 1.1 and 1.2

Before proving the theorems we establish the following :

Lemma 1 — If $\phi_k(x), = k = \overline{0, n}$ are the fundamental polynomials of Lagrange interpolation defined by (1.2), then we have

$$\sum_{k=1}^{n-1} \phi_k^2(x) \leqslant 1, \quad |\phi_k(x)| \leqslant 1, \; k = \overline{1, n-1} \qquad ...(2.1)$$

and

$$\sum_{k=1}^{n-1} |\phi_k^m(x)| \leqslant 1, \; m = 3, 4, \ldots . \qquad ...(2.2)$$

Proof : Let us consider the Hermite-Fejér interpolation process $H_n(f, x, \pi) \equiv H_n(f, \pi)$ built on the node system π, then as is well-known, $H_n(f, \pi)$ is given by[6].

$$H_n(f, \pi) = f(1)\left\{1 + (1-x)\frac{n(n+1)}{2}\right\}\phi_0^2(x) + f(-1) \times \left\{1 + (1+x)\frac{n(n+1)}{2}\right\}\phi_n^2(x) + \sum_{k=1}^{n-1} f(x_k)\,\phi_k^2(x). \qquad ...(2.3)$$

The identity by taking $f \equiv 1$ yields

$$\sum_{k=1}^{n-1} \phi_k^2(x) = 1 - \frac{2(1+x^2)\,P_n'^2(x)}{\{n(n+1)\}^2} - \frac{(1-x^2)\,P_n'^2(x)}{n(n+1)}. \qquad ...(2.4)$$

Owing to

$$(1-x^2)^{1/2} |P_n'(x)| \leqslant n, \; x \in [-1, 1] \qquad ...(2.5)$$

we have from (2.4) the consequence of (2.1). From (2.1) we easily conclude the other parts of the lemma.

The essential role is played by

Lemma 2—Let $k_1, k_2, \ldots, k_{2m}$ be distinct integers between 1 and $n-1$, we, then, have

$$\left| \int_{-1}^{1} \phi_{k_1}(x)\, \phi_{k_2}(x) \ldots \phi_{k_{2m}}(x)\, dx \right| = O\left(\frac{1}{(n+1)^m}\right), \quad m > 2 \qquad \ldots(2.6)$$

and

$$\left| \int_{-1}^{1} \phi_{k_1}(x)\, \phi_{k_2}(x) \ldots \phi_{k_{2m}}(x)\, (1-x^2)^{(m-1)/2}\, dx \right| = O\left(\frac{1}{(n+1)^{2m}}\right), \quad m \geqslant 1. \qquad \ldots(2.7)$$

The orders in (2.6) and (2.7) denote absolute positive constants.

PROOF : We have from Szegö (1959) the relations

$$|(1-x^2)^{3/4} P_n'(x)| \leqslant \sqrt{8/\pi}\, n^{1/2}, \; x \in [-1, 1] \qquad \ldots(2.8)$$

and

$$n^{1/2} (1-x_k^2)^{1/4} |P_n(x_k)| \geqslant \frac{1}{\sqrt{8\pi}}, \; k = \overline{1, n-1}. \qquad \ldots(2.9)$$

From the orthogonality property of $P_n'(x)$, we have for any polynomial $R_{n-2}(x)$ of degree $\leqslant n-2$

$$\int_{-1}^{1} \pi_{n+1}(x)\, R_{n-2}(x)\, dx = 0. \qquad \ldots(2.10)$$

It is easy to verify the following integrals

$$\int_{-1}^{1} \pi_{n+1}(x)\, P_n'(x)\, dx = \frac{2n(n+1)}{2n+1} \qquad \ldots(2.11)$$

$$\int_{-1}^{1} \pi_{n+1}^2(x)\, dx = \frac{4n^2 (n+1)^2}{(2n-1)(2n+1)(2n+3)}. \qquad \ldots(2.12)$$

Taking into account (2.5), (2.8)-(2.12), we have

$$\left| \int_{-1}^{1} \phi_{k_1}(x)\, \phi_{k_2}(x) \ldots \phi_{k_{2m}}(x)\, dx \right|$$

$$= O\left[\int_{-1}^{1} \frac{(1-x^2)^{m+1} P_n'^2(x)\, dx}{(n(n+1)^m P_n(x_{k_1}) P_n(x_{k_2}) \ldots P_n(x_{k_{2m}}) (x-x_{k_1}) \ldots (x-x_{k_{2m}})} \right]$$

$$= O\left(\frac{1}{(n+1)^m}\right). \qquad \ldots(2.13)$$

Observe that

$$\frac{(1-x^2)^m P_n'(x)}{(x-x_{k_1})(x-x_{k_2}) \ldots (x-x_{k_{2m}})} = (-1)^m P_n'(x) + q_{n-2}(x) \qquad \ldots(2.14)$$

where $q_{n-2}(x)$ is a polynomial of degree $\leqslant n-2$.

To prove the second part we make use of (2.8), (2.9), (2.14), (2.12)-(2.10).

In what follows we prove

Lemma 3 — Let $f \in C$, then, we have

$$\lim_{n\to\infty} \int_{-1}^{1} [F_n(f, \pi) - f]^2 \, dx = 0 \qquad \ldots(2.15)$$

which is Theorem 1.2 for $\rho = 1$ and Theorem 1.1 for $p = 2$.

PROOF: A well-known theorem of Gopengauz[8] states that there exists a sequence of polynomials $Q_n(x)$ of degree $\leqslant n-1$ such that

$$|g_n(x)| = |f(x) - Q_n(x)| \leqslant C\omega_f\left(\frac{\sqrt{1-x^2}}{n}\right) \qquad \ldots(2.16)$$

where C is an absolute positive constant. On account of (2.1), (2.2) (2.7) ($m = 1$) and (2.16) we obtain

$$\int_{-1}^{1} F_n^2(g_n, \pi)\, dx = \int_{-1}^{1} \sum_{k=1}^{n-1} g_n(x_k)\, \phi_k^2(x) + \sum\sum_{k=j} g_n(x_k)\, g_n(x_j)$$

$$\times \int_{-1}^{1} \phi_k(x)\, \phi_j(x)\, dx$$

$$\leqslant C \max g_n^2(x) = o(1). \qquad \ldots(2.17)$$

Owing to (2.17) and (2.16) we obtain

$$\int_{-1}^{1} [F_n(f, \pi) - f]^2 \, dx \leqslant 2\int_{-1}^{1} F_n^2(g_n, \pi)\, dx + 2\int_{-1}^{1} g_n^2(x) = o(1) \qquad \ldots(2.18)$$

which proves (2.15).

We can easly conclude from (2.15) that

$$\int_{-1}^{1} |F(g_n, \pi)|\, dx = o(1). \qquad \ldots(2.19)$$

We now indicate that we shall follow the technique employed by Erdös and Feldheim[5] viz. we confine ourselves to proving the theorems for even values of p only and the rest will follow from Minkovski-Hölder's inequality. To illustrate the method we limit the cases for $p=4$ and $\rho = 2$ respectively. Thus we need to show that for $f \in C$

$$\lim_{n\to\infty} \int_{-1}^{1} [F_n(f, \pi) - f]^4\, dx = 0 \text{ if (1.4) is fulfilled} \qquad \ldots(2.20)$$

and

$$\lim_{n\to\infty} \int_{-1}^{1} [F_n(f, \pi) - f]^4 \sqrt{1-x^2}\, dx = 0. \qquad \ldots(2.21)$$

Now on account of (2.18) it is seen that (2.20) and (2.21) are valid if we are able to show that

$$\lim_{n\to\infty} \int_{-1}^{1} | F_n^4 (g_n, \pi) | dx = 0 \qquad ...(2.22)$$

$$\lim_{n\to\infty} \int_{-1}^{1} | F_n^3 (g_n, \pi) | dx = 0 \qquad ...(2.23)$$

if (1.4) is fulfilled, and

$$\lim_{n\to\infty} \int_{-1}^{1} F_n^4 (g_n, \pi) \sqrt{1 - x^2}\, dx = 0 \qquad ...(2.24)$$

$$\lim_{n\to\infty} \int_{-1}^{1} F_n^3 (g_n, \pi)\, 4 \sqrt{1 - x^2}\, dx = 0. \qquad ...(2.25)$$

The relations (2.25) and (2.23) are easy consequences of (2.24) and (2.22). From (1.1), (1.2) and (2.16) we obtain

$$\begin{aligned} F_n^4 (g_n, \pi) = {} & \sum_{k=1}^{n-1} g_n (x_k)\, \phi_k^4 (x) + \sum\sum_{k\neq j} g_n^3 (x_k)\ \phi_k^3 (x)\, g_n (x_j)\, \phi_j^3 (x) \\ & + \sum\sum\sum_{k\neq j\neq} g_n^2 (x_k)\, g_n (x_j)\, g_n (x_l)\ \phi_k^2 (x)\, \phi_j (x)\, \phi_l (x) \\ & + \sum\sum\sum\sum_{k\neq j\neq i\neq s} g_n(x_k)\, g_n(x_j)\, g_n(x_l)\, g_n(x_s)\, \phi_k(x)\, \phi_j(x)\, \phi_l(x)\, \phi_s(x) \\ & + \sum\sum_{k\neq j} g_n^2 (x_k)\, g_n^2 (x_j)\, \phi_k^2 (x)\, \phi_j^2 (x) = \sum_{r=1}^{5} S_r (x) \text{ say.} \end{aligned} \qquad ...(2.26)$$

From (2.2) and (2.16) we obtain

$$\int_{-1}^{1} S_1 (x)\, dx = o (1). \qquad ...(2.27)$$

Next, observe that

$$S_2 = \Big(\sum_{k=1}^{n-1} g_n^3 (x_k)\, \phi_k^3 (x)\Big)\Big(\sum_{k=1}^{n-1} g_n (x_k)\, \phi_k (x)\Big) - S_1 (x).$$

Owing to Lemma 1, (2.16), (2.19) and (2.27) we, therefore, obtain

$$\int_{-1}^{-1} | S_2 (x) |\ dx = o (1). \qquad ...(2.28)$$

From (2.16) and (2.7) we have

$$\int_{-1}^{1} | S_4 (x) |\ \sqrt{1-x^2}\, dx = o (1). \qquad ...(2.29)$$

Again from (2.6) and (2.16) we get

$$\int_{-1}^{1} | S_4(x) | dx = o(1) \quad \text{with (1.4).} \qquad \ldots(2.30)$$

It is seen that

$$S_3(x) = [\sum_{k=1}^{n-1} g_n^2(x_k) \phi_k^2(x)] [F_n^2(g_n, \pi) - \sum_{j=1}^{n-1} g_n^2(x_j) \phi_j^2(x)] - S_2(x)$$

which, in turn, yields

$$\int_{-1}^{1} | S_3(x) | dx = o(1) \qquad \ldots(2.31)$$

where we have used the relations (2.16), (2.18), (2.27) and Lemma 1. Lastly, noting that

$$S_5(x) = (\sum_{k=1}^{n-1} g_n^2(x_k) \phi_k^2(x)) - S_1(x)$$

and using Lemma 1., (2.16) and (2.27) we obtain

$$\int_{-1}^{1} | S_5(x) | dx = o(1). \qquad \ldots(2.32)$$

Combining (2.32)-(2.30), (2.28) and (2.27) we obtain (2.22). In the same way taking into account the relations (2.32), (2.31), (2.29)-(2.27) we obtain (2.24). This completes the proof.

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