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GRAPHS WHOSE r-NEIGHBOURHOODS FORM CONFORMAL HYPERGRAPHS

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Given any integer $r \geqslant 1$, we characterize the graphs whose open r-neighbourhoods form a conformal hypergraph. For $r \geqslant 2$ such graphs are found to be comprised of only disjoint edges as components, and for $r = 1$ the triangle and the hexagon constitute the set of minimal forbidden subgraphs for these graphs.

§1. For all terminology and notation in graph theory and hypergraph theory, we shall refer to Harary[4] and Berge[2] respectively.

§2. Given a graph $G = (V, E)$, a positive integer r and a vertex u of G, the 'r-neighbourhood' of u, denoted $C_r(u)$, is defined to be the set of vertices of G which are at distance at most r from u, where distance between two vertices in a given connected component of G is defined to be the number of edges in a shortest path connecting the two vertices in G. The set $N_r(u) = C_r(u) - \{u\}$ is called the 'open' r-neighbourhood of u in G. Let $\mathscr{L}_r = \{C_r(u) / u \in V(G)\}$ and $\mathscr{N}_r = \{N_r(u) / u \in V(G)\}$. Then the hypergraphs $\mathscr{L}_r(G)$ and $\mathscr{N}_r(G)$ are called 'r-neighbourhood' and 'open r-neighbourhood hypergraphs' of G, respectively. Given any property of hypergraphs, say P, one may then be interested to characterize the graphs G for which $\mathscr{L}_r(G)$, or $\mathscr{N}_r(G)$, has property P (*e.g. see* Berge[3], p. 278). The aim of the present note is to attempt one instance of this problem for open r-neighbourhood hypergraphs.

§3. Given a hypergraph $H = (X, \mathscr{E})$, the graph $H_{(2)} = (X, \mathscr{E}')$ with $uv \in \mathscr{E}' \Leftrightarrow u, v \in E$ for some $E \in \mathscr{E}$ is called the '2-section' of H. One calls H 'conformal, if every 'clique' (i.e. a set of vertices any two of which are contained in an edge) of $H_{(2)}$ is contained in an edge of H (*cf.* Berge[2], p. 396). Our primary concern in this paper is to determine the structure of the graphs whose open r-neighbourhood hypergraphs are conformal. Some preliminary observations towards this end are stated in the following proposition where by a section of a hypergraph $H = (X, \mathscr{E})$ we mean a hypergraph $H' = (X, \mathscr{E}')$ such that for every $E' \in \mathscr{E}'$ there exists $E \in \mathscr{E}$ with $E' \subseteq E$. The results listed in this proposition are mostly immediate consequences of

properties of the distance metric d on the vertex set of a graph, except for (iii) which follows from (ii) and a theorem of Berge[2] (p. 397).

Proposition 1 — Let G be any graph and r be any positive integer. Then the following assertions hold.

(i) $u \in N_r(v) \Leftrightarrow v \in N_r(u)$ for $u, v \in V(G)$

(ii) $\mathscr{H}_r(G)$ is self-dual

(iii) $\mathscr{H}_r(G)$ is conformal $\Leftrightarrow$ $\mathscr{H}_r$ has the Helly property; that is for every subfamily of $\mathscr{H}_r$ having pairwise intersection property (i.e. any two members of it have nonempty intersection) the whole intersection property holds (i.e. the intersection of all the members of the subfamily is nonempty)

(iv) for $r \geqslant 2$, $N_{r-1}(u) \subseteq N_r(u)$ for each $u \in V(G)$

(v) for $r \geqslant 2$, $\mathscr{H}_{r-1}(G)$ is a section of $\mathscr{H}_r(G)$.

Proposition 2 — For any graph G and any integer $r \geqslant 1$, if $\mathscr{H}_r(G)$ is conformal then G has no cycle of length $\leqslant 3r$.

PROOF : Suppose that $\mathscr{H}_r(G)$ is conformal and G has a cycle Z of length at most $3r$. Then, we may choose any three vertices u_1, u_2, u_3 on Z so that $d(u_1, u_2) \leqslant r$, $d(u_2, u_3) \leqslant r$ and $d(u_1, u_3) \leqslant r$. This implies $u_1, u_2 \in N_r(u_3)$, $u_2, u_3 \in N_r(u_1)$ and $u_1, u_3 \in N_r(u_2)$. Hence, $\{u_1, u_2, u_3\}$ would be a clique in the 2-section of $\mathscr{H}_r(G)$ such that $\{N_r(u_1), N_r(u_2), N_r(u_3)\}$ has pairwise intersection property. Let Q be a clique of maximum size in the 2-section of $\mathscr{H}_r(G)$ such that the family $\{\mathscr{H}_r(u) | u \in Q\}$ has pairwise intersection property. As seen above, $|Q| \geqslant 3$. Since $\mathscr{H}_r(G)$ is conformal we see, from Proposition 1 (iii), that $\mathscr{H}_r$ must satisfy Helly property and hence as $\{N_r(u) | u \in Q\}$ has pairwise intersection property we must have a vertex $x \in \bigcap_{u \in Q} N_r(u)$. Then by Proposition 1 (i), $u \in N_r(x)$ for every $u \in Q$ so that $Q \subseteq N_r(x)$. Since $x \notin N_r(x)$ we have $x \notin Q$. Let $Q_x = \{x\} \cup Q$. Since $|Q| \geqslant 3$, for any given $u \in Q$ there exists $u' \in Q - \{u\}$ such that $x, u \in N_r(u')$ so that xu is an edge in the 2-section of $\mathscr{H}_r(G)$. Thus, x is adjacent to every vertex of Q in the 2-section of $\mathscr{H}_r(G)$ and hence Q_x is a clique of $(\mathscr{H}_r(G))_{(2)}$ such that $\{N_r(y) | y \in Q_x\}$ has pairwise intersection property. But, this contradicts the maximality of Q as $|Q_x| = |Q| + 1$. Hence the assertion follows. □

Proposition 3—For any graph G, if $\mathscr{H}_r(G)$ is conformal for an integer $r \geqslant 2$ then no two edges of G are incident.

PROOF : Suppose that uv and vw are two incident edges in G and that $\mathscr{H}_r(G)$ is conformal for some $r \geqslant 2$. Then, it is easy to see that $v, w \in N_r(u)$, $u, w \in N_r(v)$ and

$u, v \in N_r(w)$, and hence $\{u, v, w\}$ is a clique in the 2-section of $\mathcal{H}_r(G)$. Since $N_r(G)$ is conformal it follows that there must exist $x \in V(G)$ such that $\{u, v, w\} \subseteq N_r(x)$. This implies that the subgraph $< N_r(x) >$ induced by $N_r(x)$ in G has a cycle of length $\leqslant 2r + 1 < 3r$ as $r \geqslant 2$. But, this is contrary to Proposition 2. □

I had announced Proposition 2 for the case $r = 1$ in a seminar on Graph Theory held in our Institute during December 1981, under the auspices of the 47th Conference of the Indian Mathematical Society (*also see* Acharya[1]) and had conjectured that C_3 (the triangle) and C_6 (the 'hexagon', or the cycle of length 6) comprise the set of minimal forbidden subgraphs for graphs whose open neighbourhood hypergraphs are conformal. I shall now sketch below a proof of this conjecture. For convenience, $N_1(u)$ and $\mathcal{H}_1(G)$ will henceforth be written simply as $N(u)$ and $\mathcal{H}(G)$, respectively.

Proposition 4—The open neighbourhood hypergraph $\mathcal{H}(G)$ of a connected graph G is conformal if and only if G has neither C_3 nor C_6 as an induced subgraph.

PROOF : *Necessity* : Let $\mathcal{H}(G)$ be conformal. Then, that G has no C_3 as an induced subgraph follows from Proposition 2. Hence, suppose that G has $C_6 = (u_1, u_2, u_3, u_4, u_5, u_6, u_1)$ as an induceds ubgraph. Then $u_2, u_6 \in N(u_1)$, $u_2, u_4 \in N(u_3)$, $u_4, u_6 \in N(u_5)$ and $u_1, u_3 \in N(u_2)$, $u_3, u_5 \in N(u_4)$, $u_5, u_1 \in N(u_6)$. Thus, $\{u_1, u_3, u_5\}$ is a clique in the 2-section of $\mathcal{H}(G)$ such that $\{N(u_1), N(u_3), N(u_5)\}$ has pairwise intersection property in G. Hence, we let Q be a clique of maximum cardinality in the 2-section of $\mathcal{H}(G)$ such that $\{N(u)/u \in Q\}$ has pairwise intersection property in G. Then as in the proof of Proposition 2 using conformality of $\mathcal{H}(G)$ we are led to a contradiction to the maximality of Q.

Sufficiency : Suppose that G has no induced triangles or hexagons but that $\mathcal{H}(G)$ is not conformal. Then, by applying a theorem of Gilmore (cf. Berge[2], p. 396) to $\mathcal{H}(G)$ it follows that there must exist $u, v, w \in V(G)$ such that $\{N(u), N(v), N(w)\}$ has pairwise intersection property while the set $N(u, v, w) = (N(u) \cap N(v)) \cup (N(v) \cap N(w)) \cup (N(w) \cap N(u))$ is not contained in the open neighbourhood of any vertex of G. Let $x \in N(u) \cap N(v)$, $y \in N(v) \cap N(w)$ and $z \in N(w) \cap N(u)$. Since G has no triangles, it is easy to see that $\{u, v, w\}$ and $\{x, y, z\}$ must be independent sets in G (as also the open neighbourhoods of the vertices in them). Since G does not contain an induced hexagon as well, we see that at least one of the pairs $\{u, y\}$, $\{v, z\}$ and $\{w, x\}$ must be an edge of G; say $uy \in E(G)$. Then, $\{x, y, z\} \subseteq N(u)$. Since $N(u, v, w) \not\subseteq N(u)$ there must exist $y' \in (N(v) \cup N(w)) - \{y\}$ such that $uy' \notin E(G)$. Now, if neither $\{w, x\}$ nor $\{v, z\}$ is an edge of G, then $< \{x, v, y', w, z, u\} >$ is a hexagon, so assume $vz \in E(G)$. This yields $\{x, y, z\} \subseteq N(v)$; since $N(u, v, w) \not\subseteq N(v)$, there must exist $z' \in (N(u) \cup N(w)) - \{z\}$ such that $vz' \notin E(G)$. Then if $wx \notin E(G)$ we obtain the induced hexagon $< \{x, v, y', w, z', u\} >$ in G. Thus, $wx \in E(G)$ and hence $\{x, y, z\} \subseteq N(w)$. Since $x \in N(u) \cap N(v)$, $y \in N(v) \cap N(w)$ and $z \in N(w) \cap N(u)$ are arbitrary by choice we see that $N(u, v, w)$ is contained in each of $N(u)$, $N(v)$ and $N(w)$, a contradiction. Thus, $\mathcal{H}(G)$ must be conformal. □

We have thus proved the following characterization.

Theorem—For any isolate-free graph G and any integer $r \geqslant 1$, $\mathscr{H}_r(G)$ is conformal if and only if (i) $r = 1$ and G has neither C_3 nor C_6 as an induced subgraph, and (ii) $r \geqslant 2$ and no two edges of G are incident.

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