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LINEARITY IN A PLANAR TERNARY RING WITH ZERO OF A PROJECTIVE PLANE

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In this paper a weak Desargues property, the validity of which is equivalent to the validity of the linearity of a coordinatizing planar ternary ring with zero (PTRZ) of a projective plane is given.

1. PRELIMINARIES

Let π be a projective plane and X, Y, O any three distinct non-collinear points in π. Consider a non-empty set S with a fixed element 0 and bijections $a \to [a]$, $b \to [0, b]$, $m \to (m)$ of S onto the set of all lines $\neq XY$ through Y and through X, the set of all points $\neq Y$ on XY respectively, such that $OY = [0]$, $OX = [0, 0]$, $X = (0)$. With an element $\infty \notin S$, the line XY is denoted by $[\infty]$ and the point Y by (∞). Then a bijection of $S \times S$ onto the set of all points P not on XY is determined by $PY = [a]$, $PX = [0, b]$ and P will be denoted by (a, b), especially therefore O by $(0, 0)$. Dually generalizing the symbol $[0, b]$, a line l not through Y will be denoted by $[m, d]$, where $l \cap XY = (m)$, $l \cap OY = (0, d)$.

Define the ternary operation $\langle\ \rangle$ on S by : the point (x, y) is on the line $[m, b]$ if and only if $\langle x, m, b \rangle = y$. Then one can see that $(S, \langle\ \rangle)$ is a PTRZ, that means, a ternary system satisfying the following axioms[1].

T_1. $\langle a, b, x \rangle = c$ has a unique solution for $x \in S$.

T_2. $\langle x, a, b \rangle = \langle x, c, d \rangle$, $a \neq c$ has a unique solution for $x \in S$.

T_3. $\langle a, x, y \rangle = b$, $\langle c, x, y \rangle = d$, $a \neq c$ has a unique solution pair (x, y), $x, y \in S$.

T_4. $\langle a, 0, c \rangle = \langle 0, a, c \rangle = c$.

From T_3 and T_4 we have

Z_1. $\langle a, x, b \rangle = c$, $a \neq 0$ has a unique solution for $x \in S$.

Thus in a PTRZ, for every $a \neq 0$, there is an element e_a uniquely determined by $\langle a, e_a, 0 \rangle = a$ and we define $e_0 = 0$ and

$$a + b = \langle a, e_a, b \rangle; \; a \cdot b = \langle a, b, 0 \rangle.$$

Definition—A *PTRZ* $(S, \langle\ \rangle)$ is called linear, if for all $a, m, b \in S$, $\langle a, m, b \rangle = a.\, m + b$.

Let P, Q, R be any three distinct points and m a line in π such that $P, R \in m$, $Q \notin m$. Then the $(P, m; Q, R)$—Desargues property for triangles is the (P, m)-Desargues property where

(i) on vertex of a triangle coincides with Q and

(ii) the pair of corresponding sides not containing Q and its corresponding vertex intersect at R.

2. Main Theorem

Theorem—Let π be a projective plane and $(S, \langle\ \rangle)$ its coordinatizing PTRZ relative to the reference points X, Y, O. Then $(S, \langle\ \rangle)$ is linear if and only if the $(Y, XY; O, X)$—Desargues property holds in π.

Proof : Suppose $(S, \langle\ \rangle)$ is linear and consider the triangles ABC, $A'B'C'$ which are such that AA', BB', CC' pass through Y; $C = O$; $AB \cap A'B' = X$ and $AC \cap A'C' = M$ where $M \in XY$. Then $AA' = [a]$, $BB' = [c]$, $C' = (0, b)$ for some $a, b, c \in S$. Let $M = (m)$ for some $m \in S$.

From $AC \cap A'C' = (m)$, it follows that $A = (a, am)$ and $A' = (a, \langle a, m, b \rangle)$. $AB \cap A'B' = X$ implies that $B = (c, am)$ and $B' = (c, \langle a, m, b \rangle)$. Let $BC \cap XY = (n)$ and $B'C' \cap XY = (n')$ for some $n, n' \in S$. Then $BC = [n, 0]$ and $B'C' = [n', b]$. Now $B = (c, am) \in [n, 0]$ implies $cn = am$. Again $B' = (c, \langle a, m, b \rangle) \in [n', b]$ implies $\langle c, n', b \rangle = \langle a, m, b \rangle = am + b = cn + b = \langle c, n, b \rangle$ which implies $n = n'$ by Z_1. Therefore $BC \cap B'C' = (n) \in XY$.

Conversely, assume that $(Y, XY; O, X)$—Desargues property holds in π. Consider the triangles ABC, $A'B'C'$ where $C = (0, 0)$; $C' = (0, b)$; $A = [a] \cap [m, 0] = (a, am)$; $B = [am] \cap [e_{am}, 0] = (am, am)$; $B' = [am] \cap [e_{am}, b] = (am, \langle am, e_{am}, b \rangle) = (am, am + b)$; $A' = [a] \cap [0, am + b] = (a, am + b)$. It is evident that AA', BB', CC' pass through Y; $C = O$; $AB \cap A'B' = X$ and $BC \cap B'C' = (e_{am})$. The validity of $(Y, XY; O, X)$—Desargues property applied to the above triangles yields $AC \cap A'C' \in XY$. But $AC \cap XY = (m)$ so that $A'C' \cap XY = (m)$. Thus $A'C' = [m, b]$ and $A' = (a, am + b) \in [m, b]$ implies $\langle a, m, b \rangle = am + b$.

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