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ABSOLUTE NÖRLUND SUMMABILITY FACTORS FOR FOURIER SERIES

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In this paper, two theorems are proved in which neither the conditions imposed upon the generating functions of the Fourier series nor the absolute summability factors can be improved in certain senses. Theorem 1 improves an earlier result due to **Kanno**[6], while Theorem 2 yields some new results.

1. DEFINITIONS AND NOTATIONS

Let Σa_n be a given infinite series with the sequence of partial sums (s_n) and (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \ldots + p_n;\ P_{-1} = p_{-1} = 0.$$

The sequence-to-sequence transformation

$$t_n = \sum_{m=0}^{n} P_m\, a_{n-m}/P_n;\ P_n \neq 0$$

defines the sequence (t_n) of Nörlund means[13] of the sequence (s_n), generated by the sequence of coefficients (p_n). The series Σa_n is said to be absolutely summable[11] (N, p_n) or $|N, p_n|$, if the sequence (t_n) is of bounded variation (or $(t_n) \in bv$), i.e. the infinite series

$$\sum_n \left| t_n - t_{n-1} \right| = \sum_n \frac{1}{P_n P_{n-1}} \left| \sum_{k=0}^{n-1} (P_n p_k - P_k p_n)\, a_{n-k} \right| < \infty.$$

The necessary and sufficient conditions for the regularity of $|N, p_n|$ method are $p_n = o(|P_n|)$, as $n \to \infty$, and

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$$\sum_{n=k}^{\infty} \left| \frac{P_{n-k}}{P_n} - \frac{P_{n+1-k}}{P_{n+1}} \right| < K$$

K being independent of k, $k = 1, 2, \ldots$.

We define $(C, 1)\ (N, p_n)$—mean of (s_n) as the $(C, 1)$ - mean of (t_n). Thus the $(C, 1)\ (N, p_n)$—mean of (s_n) is

$$t_n' = \sum_{m=0}^{n} t_m/(n+1).$$

The series Σa_n is said to be absolutely summable $(C, 1)\ (N, p_n)$ or summable $|\ (C, 1)\ (N, p_n)\ |$, if

$$\sum_n \left| t_n' - t_{n-1}' \right| = \sum_n \frac{1}{n(n+1)} \left| \sum_{m=1}^{n} \frac{m}{P_m P_{m-1}} \sum_{k=0}^{m-1} (P_m p_k - p_m P_k)\, a_{m-k} \right|$$

is finite

Let f be 2π-periodic and L-integrable over $(-\pi, \pi)$. We assume without loss of generality that the Fourier series of f is given by

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt = \sum_{n=1}^{\infty} A_n(t).$$

We write for $c \geqslant 1$ and fixed real x :

$$\phi(t) = \frac{1}{2} \{f(x+t) + f(x-t)\} \qquad \ldots(1.1)$$

$$\phi_1(t) = (1/t) \int_0^t \phi(u)\, du \qquad \ldots(1.2)$$

$$R(t) = \phi(t) - \phi_1(t) \qquad \ldots(1.3)$$

$$R_1(t) = \phi_1(t) - (1/t) \int_0^t \phi_1(u)\, du \qquad \ldots(1.4)$$

$$h(t, c) = \left(\log \log \frac{1}{t} \right)^c \qquad \ldots(1.5)$$

$$y_n = 1/\log(n+1) \quad (n \geqslant 1) \qquad \ldots(1.6)$$

$$J_n = \sum_{k=n}^{\infty} (kP_k)^{-1} y_{k+1} \qquad \ldots(1.7)$$

$$g(n, k) = P_{n-k}/P_n - P_{n-k-1}/P_{n-1}, \ (n \geqslant 1) \qquad \ldots(1.8)$$

$$C_n = n\, p_n/P_n \qquad \ldots(1.9)$$

$$\Delta_n\, t_{n,k} = t_{n,k} - t_{n+1,k} \qquad \ldots(1.10)$$

for any sequence $(t_{n,k})$.

Throughout T denotes the integral part of $(r/2t)$ for some constant r.

2. Introduction

Varshney[15] proved the following theorem :

Theorem A—Let

$$\phi(t) \in BV(0, \pi). \qquad \ldots(2.1)$$

Then

$$\sum_{n=1}^{\infty} A_n(x)\, y_n \in \left|N, \frac{1}{n+1}\right|.$$

This result was extended by Singh[14] and Lal[8] in different directions. Kanno[6] extended Theorem *A* in the following form :

Theorem B— Let (p_n) be a positive non-increasing sequence such that $P_n \to \infty$, as $n \to \infty$, and let

$$(C_n/y_{n+1}) \in bv \qquad \ldots(2.2)$$

$$\left(\frac{1}{P_n}\sum_{k=1}^{n}\frac{P_k\, y_k}{k}\right) \in bv. \qquad \ldots(2.3)$$

Then (2.1) implies that

$$\sum_{n=1}^{\infty} A_n(x)\, y_n \in |N, p_n|.$$

Lal[9] has obtained another generalisation of Theorem *A* but no one has discussed the following aspects of Theorem *A* :

(i) Is it possible to weaken the hypothesis (2.1) in Theorem *A* ?

(ii) Can the summability factor (y_n) in Theorem *A* be bettered ?

In this paper, we first give an answer to question (i) partially in affiirmative and then we show that the factor (y_n) cannot be bettered. In this context, we prove the following :

Theorem 1—Let (p_n) be a non-negative and non-increasing sequence such that

$$P_n J_n = O\{\log\log n)^c\}. \qquad \ldots(2.4)$$

Then, in order that

$$\sum_{n=1}^{\infty} A_n(x)\, y_n \in |N, p_n|$$

whenever,

$$R(t)\, h(t, c) \in BV(0, \pi) \qquad \ldots(2.5)$$

it is necessary and sufficient that

$$c > 1. \qquad \ldots(2.6)$$

Further, the factor (y_n) cannot be replaced by (y_n^{β}) $(0 < \beta < 1)$.

Before discussing the various relations between the hypotheses of Theorem *B* and Theorem 1, we state the following theorem, which we intend to prove for the class of those functions for which

$$(*) \lim_{\epsilon \to 0} \int_{\epsilon}^{t} \phi_1(u)\, du$$

exists (in the sense of Cauchy) :

Theorem 2—Let (p_n) be a non-negative and non-increasing such that

$$(C_n) \in bv \qquad \ldots(2.7)$$

and

$$P_n J_n = O(1). \qquad \ldots(2.8)$$

Then, in order that

$$\sum_{n=1}^{\infty} A_n(x)\, y_n \in |(C, 1)\, N, p_n)|$$

whenever, $R_1(t)\, h(t, c) \in BV(0, \pi)$, it is necessary and sufficient that (2.6) should hold. Further, (y_n) cannot be replaced by (y_n^{β}) $(0 < \beta < 1)$.

Remarks : Suppose that, in some right hand neighbourhood of the origin

$$\phi(t) = (t \log(1/t))^{-2}.$$

Then $\phi(t)$ is integrable in a neighbourhood of the origin but $\phi_1(t)$ is not integrable, since for sufficiently small t

$$\phi_1(t) = (t \log(1/t))^{-1}.$$

Consequently $R(t)$ is not integrable and hence $R_1(t)$ is not defined. Thus $R_1(t)$ does not exist for all L-integrable functions and hence the assumption of (*) is needed.

It is known (Chandra[4]) that (2.1) holds if and only if

$$(a)\ R(t) \in BV(0, \pi); \quad (b)\ t^{-1}R(t) \in L(0, \pi) \qquad \ldots(2.9)$$

and the condition (2.5) does not imply (2.9b). Therefore (2.5) is a condition not stronger than (2.1). It may further be observed that the class of functions satisfying (2.5) not only includes the functions satisfying (2.1) but also includes even some functions which are not bounded (see Chandra[3]). This gives an answer to (i) partially in affirmative. Now, as regards the relationship between (2.3) and (2.4), we first observe that (2.3) implies that

$$\sum_{k=1}^{n} P_k y_k/k = O(P_n) \qquad \ldots(2.10)$$

and the equivalence of (2.10) and (2.8) follows from Lemma 4 of this paper. Thus we remark that in Theorem 1 the extra hypothesis (2.2) of Theorem B has been dropt and (2.3) is replaced by a more general condition (2.4). On the other hand, we prove Theorem 2 under (2.7) and (2.8).

3. LEMMAS

We shall require the following lemmas in the proof of the theorems :

Lemma 1—For $k > \pi e^2$, $\alpha \geqslant 0$ and $\beta > 1$

$$\int_0^t \frac{\sin nu}{u\left(\log \frac{k}{u}\right)^{\alpha} h(u, \beta)}\, du = O\{y_n^{\alpha}/h(1/(n+2), \beta)\}$$

uniformly in $0 < t \leqslant \pi$.

The case $\alpha = 0$ is due to Chandra[1]; (3.2). The general case may be dealt similarly.

Lemma 2—For $k \geqslant \pi e^2$ and all real β,

$$\int_0^{\pi} \left(\log_2 \frac{k}{t}\right)^{\beta} \frac{\sin nt}{t}\, dt \sim \frac{\pi}{2} (\log_2 (n+2))^{\beta}.$$

This is essentially contained in Theorem 2.24 of Zygmund[16] (p. 190). However for an explicit proof, see Chandra[5].

Lemma 3—Let (p_n) be a non-negative and non-increasing sequence. Then for $0 \leqslant a \leqslant b \leqslant \infty$ and any n

$$\sum_{k=a}^{b} p_k \exp(i\,(n-k)\,t) \leqslant K\,P_T$$

uniformly in $0 < t \leqslant \pi$.

See Lemma (5.11) of McFadden[10].

Lemma 4—(2.10) implies and is implied by (2.8) whenever $p_n \geqslant 0$.

PROOF : (i) Let (2.10) hold. Then for large N, we have by Abel's transformation

$$\sum_{k=n}^{N} \frac{y_k}{k\,P_k} \leqslant 2 \sum_{k=n}^{N-1} \frac{p_{k+1}}{P_k^2\,P_{k+1}} \sum_{m=n}^{K} \frac{P_m\,y_m}{m} + \frac{1}{P_N^2} \sum_{m=n}^{N} \frac{P_m\,y_m}{m}$$

$$= O(1) \sum_{k=n}^{N-1} \Delta\left(\frac{1}{P_k}\right) + O\left(\frac{1}{P_n}\right)$$

$$= O(1/P_n)$$

since (2.10) implies that

$$\sum_{m=n}^{k} \frac{P_m\,y_m}{m} = O(P_k) \text{ for } k \geqslant n.$$

Hence (2.8) holds.

(ii) Let (2.8) hold. Then

$$\sum_{k=1}^{n} k^{-1}\,P_k\,y_{k+1} = \sum_{k=1}^{n} P_k^2\,\Delta\,J_k$$

$$= \sum_{k=1}^{n} (P_k^2 - P_{k-1}^2)\,J_k - P_n^2\,J_{n+1}$$

$$\leqslant \sum_{k=1}^{n} (P_k^2 - P_{k-1}^2)\,J_k$$

$$\leqslant 2 \sum_{k=1}^{n} p_k\,J_k\,P_k$$

$$= O(P_n)$$

by (2.8).

This completes the proof of the lemma.

Lemma 5—Let (p_n) be non-negative and non-increasing. Then, for all $k \geqslant 0$ and $1 \leqslant a \leqslant b \leqslant \infty$,

$$\sum_{n=a}^{b} g(n, k) \leqslant 1 \qquad \ldots(3.1)$$

and for any $n > 0$, $g(n, k) \geqslant 0$ so that the (N, p) method is absolutely regular.

PROOF : The validity of (3.1) may be checked by term addition and since

$$g(n, k) = \frac{p_{n-k} P_n - p_n P_{n-k}}{P_n P_{n-1}}$$

it follows that $g(n, k) \geqslant 0$. Now, in order that (N, p_n) is absolutely regular, it is necessary and sufficient that[12,7]

$$p_n = o(|P_n|), \text{ as } n \to \infty \qquad \ldots(3.2)$$

and

$$\sum_{n=k}^{\infty} |g(n, k)| < K \qquad \ldots(3.3)$$

K being independent of k, $k = 1, 2, \ldots$.

By using the fact that $(n+1) p_n \leqslant P_n$, (3.2) follows, and since $g(n, k) \geqslant 0$, the proof of (3.3) follows from (3.1).

Lemma 6—Let $H(t) \in L(0, \pi)$ be given. Then we can determine $\phi(t) \in L(0, \pi)$ such that

$$\phi(t) - \phi_1(t) = H(t). \qquad \ldots(3.4)$$

PROOF : To prove this, we define

$$\phi(t) = H(t) - \int_t^{\pi} u^{-1} H(u)\, du \qquad \ldots(3.5)$$

and show that this has the properties required. We first show that $\phi(t) \in (L\, 0, \pi)$. We are given that $H(t) \in L(0, \pi)$, so it is enough to consider the second term on the right of (3.5) :

$$\int_0^{\pi} dt \left| \int_t^{\pi} u^{-1} H(u)\, du \right| \leqslant \int_0^{\pi} dt \int_t^{\pi} u^{-1} |H(u)|\, du$$

$$= \int_0^{\pi} u^{-1} |H(u)|\, du \int_0^{u} dt$$

$$= \int_0^{\pi} |H(u)|\, du < \infty$$

which is as required. Next, since the above argument shows that the double integral exists as a Lebesgue integral, therefore we may invert the order of integration, we have from (3.5)

$$\phi_1(t) = t^{-1}\int_0^t H(u)\,du - t^{-1}\int_0^t dv \int_v^\pi u^{-1}H(u)\,du$$

$$= t^{-1}\int_0^t H(u)\,du - t^{-1}\int_0^\pi u^{-1}H(u)\,du \int_0^{\min(u,t)} dv$$

$$= t^{-1}\int_0^t H(u)\,du - t^{-1}\int_0^\pi u^{-1}H(u)\,du.\ \min(u,t)$$

$$= t^{-1}\int_0^t H(u)\,du - t^{-1}\left\{\int_0^t H(u)\,du + \int_t^\pi u^{-1}H(u)\,du.\ t\right\}$$

$$= -\int_t^\pi u^{-1}H(u)\,du.$$

Using this in (3.5), we get (3.4).

Lemma 7—Let (p_n) be non-negative and non-increasing and let $N = [\frac{1}{2}n]$, $T = [2\pi/t]$. Then uniformly in $0 < t \leqslant \pi$,

$$\sum_{k=1}^{n} p_{n-k}\,y_k \exp(ikt) = O\{t^{-1}p_N\,y_T\} + O\{P_T\,y_n\}.$$

PROOF: We first observe that $N \leqslant n-N$ and hence

$$p_{n-N} \leqslant p_N.$$

Now,

$$\sum_{k=1}^{n} p_{n-k}\,y_k \exp(ikt) = \left(\sum_{k=1}^{N} + \sum_{N+1}^{n}\right)(p_{n-k}\,y_k \exp(ikt))$$

$$= \sum\nolimits_1 + \sum\nolimits_2, \text{ say.}$$

However, by Abel's lemma

$$\left|\sum\nolimits_1\right| \leqslant p_N \max_{1\leqslant j\leqslant N}\left|\sum_{k=j}^{N} y_k \exp(ikt)\right|$$

and, by Lemma 3,

$$\left|\sum_{k=j}^{N} y_k \exp(ikt)\right| = O\{t^{-1}y_T\}.$$

Lastly, once again by Abel's lemma and Lemma 3,

$$\left|\sum_{3}\right| \leqslant y_N \max_{N+1 \leqslant N' \leqslant n} \left|\sum_{N+1}^{N'} p_{n-k} \exp(ikt)\right|$$

$$= O\{y_n P_n\}.$$

Combining the results, we get the proof of the lemma.

4. Proof of Theorem 1

4.1. *Condition* (2.6) *is sufficient*—We have (see Chandra[2], Theorem 1)

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} R(t)\, h(t, c)\, \frac{t}{h(t, c)}\, \frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right) dt$$

$$= \frac{2}{\pi}\int_0^{\pi} R(t)\, h(t, c)\, q(n,t)\, dt, \text{ say.}$$

Integration by parts and an application of Lemma 1 yield that

$$\int_0^{t} q(n, u)\, du = \frac{\sin nt}{n\, h(t, c)} + O\{n^{-1}/h(1/(n+2), c)\}$$

uniformly in $0 < t \leqslant \pi$. Thus, by integrating by parts and (2.5), we have

$$A_n(x) = d_n - \frac{2}{\pi}\int_0^{\pi} d\{R(t)\, h(t, c)\}\frac{\sin nt}{n\, h(t, c)}$$

where

$$|d_n| \leqslant \frac{M}{n\, h(1/(n+2), c)}.$$

Since, $\Sigma d_n y_n$ is absolutely convergent, therefore by absolute regularity of (N, p_n) method (see Lemma 5),

$$\Sigma d_n y_n \in |N, p_n|.$$

Thus, for the proof, it is sufficient to show that

$$\sum = \sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}}\left|\sum_{m=1}^{n} (P_n p_{n-m} - p_n P_{n-m})\frac{y_m \sin mt}{m}\right| = O\{h(t, c)\}$$

uniformly in $0 < t < \pi$.

We first note that

$$g(n, m) = \frac{P_n p_{n-m} - p_n P_{n-m}}{P_n P_{n-1}}. \qquad \ldots(4.1.1)$$

Then, we write

$$\sum = \sum_{n=1}^{2T+1} \left| \sum_{m=1}^{n} g(n, m)\, m^{-1} y_m \sin mt \right|$$

$$+ \sum_{n=2T+2}^{\infty} \left| \left(\sum_{m=1}^{T} + \sum_{T+1}^{n} \right) g(n, m)\, m^{-1} y_m \sin mt \right|$$

$$\leqslant \Sigma_1 + \Sigma_2 + \Sigma_3 \text{ say.}$$

By using $|\sin mt| \leqslant mt$, Lemma 5 and changing the order of summation, we obtain that

$$\Sigma_1 \leqslant t \sum_{m=1}^{2T+1} y_m \sum_{n=m}^{2T+1} g(n, m) = O(1).$$

And once again by Lemma 5,

$$\Sigma_2 \leqslant t \sum_{m=1}^{T} y_m \sum_{n=2T+2}^{\infty} g(n, m) = O(1).$$

Now, for $s = [\frac{1}{2} n]$, we write

$$\Sigma_3 = \sum_{n=2T+2}^{\infty} \left| \left(\sum_{m=T+1}^{s} + \sum_{s+1}^{n} \right) g(n, m)\, m^{-1} y_m \sin mt \right|$$

$$\leqslant \Sigma_{3,1} + \Sigma_{3,2} \text{, say.}$$

And, by (4.1.1), Abel's lemma and Lemma 3,

$$\Sigma_{3,2} \leqslant \sum_{n=2T+2}^{\infty} (1/P_{n-1}) \left| \sum_{m=s+1}^{n} p_{n-m}\, m^{-1} y_m \sin mt \right|$$

$$+ \sum_{n=2T+2}^{\infty} \frac{p_n}{P_n P_{n-1}} \left| \sum_{m=s+1}^{n} P_{n-m}\, m^{-1} y_m \sin mt \right|$$

$$= O(P_T) \sum_{n=2T+2}^{\infty} \frac{y_n}{n P_{n-1}} + O(t^{-1}) \sum_{n=2T+2}^{\infty} \frac{p_n y_n}{n P_n}$$

$$= O\{h(t, c)\} + O(t^{-1}) \sum_{n=2T+2}^{\infty} n^{-2}$$

$$= O\{h(t, c)\}$$

by (2.4).

Finally, to complete the proof, we show that

$$\Sigma_{3,1} = O(1). \qquad \ldots(4.1.2)$$

By Abel's transformation

$$\left|\sum_{m=T+1}^{s} g(n, m)\, m^{-1}\, y_m \sin mt\right| \leqslant \left|\sum_{m=T+1}^{s-1} \Delta_m\{g(n,m)\, m^{-1}\, y_m\} \sum_{\mu=T+1}^{m} \sin \mu t\right|$$

$$+ g(n, s)\, s^{-1}\, y_s \left|\sum_{\mu=T+1}^{s} \sin \mu t\right|$$

$$= O(t^{-1}) \sum_{m=T+1}^{s-1} \left|\Delta_m \{g(n, m)\, m^{-1}\, y_m\}\right|$$

$$+ O\{t^{-1}\, n^{-1}\, g(n, s)\}$$

where

$$\Delta_m \{g(n, m)\, m^{-1}\, y_m\} = g(n, m)\, \Delta\left(\frac{y_m}{m}\right) + \frac{y_{m+1}}{m+1}\, \Delta_m\, g(n, m)$$

$$= g(n, m)\, O\left(\frac{y_m}{m^2}\right) - \frac{y_{m+1}}{m+1}\left(\frac{p_{n-m-1}}{P_{n-1}} - \frac{p_{n-m}}{P_n}\right).$$

Therefore

$$\sum\nolimits_{3,1} = O(t^{-1}) \sum_{m=2T+2}^{\infty} n^{-1} y_n\, g(n, s) + O(t^{-1}) \sum_{n=2T+2}^{\infty} \sum_{m=T+1}^{s-1} g(n, m)\, \frac{y_m}{m^2}$$

$$+ O(t^{-1}) \sum_{n=2T+2}^{\infty} \sum_{m=T+1}^{s-1} \frac{y_{m+1}}{m+1}\left(\frac{p_{n-m-1}}{P_{n-1}} - \frac{p_{n-m}}{P_n}\right)$$

$$= \sum\nolimits_1^* + \sum\nolimits_2^* + \sum\nolimits_3^*, \text{ say.}$$

By (4.1.1) and the fact $P_n\, p_{n-s} \geqslant p_n\, P_{n-s}$, we have

$$\sum\nolimits_1^* = O(t^{-1}) \sum_{n=2T+2}^{\infty} n^{-1}\, y_n\, \frac{p_{n-s}}{P_n}$$

$$= O(t^{-1}) \sum_{n=2T+2}^{\infty} n^{-2}$$

$$= O(1).$$

And, as $N \to \infty$,

$$\sum\nolimits_2^* = O(t^{-1}) \sum_{m=T+1}^{\infty} (y_m/m^2) \sum_{n=2m}^{\infty} g(n, m)$$

$$= O(1)$$

by Lemma 5. And

$$\sum\nolimits_3^* = O(t^{-1}) \sum_{m=T+1}^{\infty} \frac{y_{m+1}}{m+1} \sum_{n=2m}^{\infty} \left(\frac{p_{n-m-1}}{P_{n-1}} - \frac{p_{n-m}}{P_n} \right)$$

$$= O(t^{-1}) \sum_{m=T+1}^{\infty} \frac{y_{m+1}\, p_{m-1}}{(m+1)\, P_{2m-1}}$$

$$= O(1).$$

Combining Σ_i^* $(i = 1, 2, 3)$, we obtain (4.1.2) and consequently it completes the proof.

4.2. *The condition* (2.6) *is necessary*—Let $\Sigma A_n(x)\, y_n \in |N, p_n|$ whenever (2.5) holds. Then, in view of Lemma 6, we may choose

$$R(t) = 1/h(t, c)$$

so that

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \frac{\cos nt}{h(t, c)}\, dt - \frac{2}{\pi} \int_0^{\pi} \frac{\sin nt}{nt\, h(t, c)}\, dt$$

$$= -\frac{2c}{n\pi} \int_0^{\pi} \frac{\sin nt}{t \log \frac{1}{t} \left(\log \log \frac{1}{t} \right)^{1+c}}\, dt$$

$$- \frac{2}{n\pi} \int_0^{\pi} \frac{\sin nt}{t\, h(t, c)}\, dt$$

$$= \alpha_n + \beta_n, \text{ say.}$$

By Lemma 1,

$$\Sigma |\alpha_n y_n| < \infty$$

and so, a fortiori, summable $|N, p_n|$. Thus, in order that $\sum_{n=1}^{\infty} A_n(x)\, y_n \in |N, p_n|$, it is necessary that

$$\sum_{n=1}^{\infty} \beta_n y_n \in |N, p_n|. \qquad \ldots(4.2.1)$$

However, by Lemma 2,

$$\beta_n \sim -\frac{1}{n(\log\log(n+2))^c}$$

and hence $\sum_{n=1}^{\infty} y_n \beta_n$ diverges to $-\infty$ for $c \leqslant 1$. Consequently it cannot be summable (and hence not absolutely summable) by any totally regular matrix transformation. Thus for the truth of (4.2.1), it is necessary that $c > 1$.

5. Proof of Theorem 2

5.1. *The condition is sufficient*—We have (Chandra[2], Theorem 1)

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} t\,R(t)\,\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt}\right)dt.$$

Now in view of the fact that

$$R_1(t) = (1/t)\int_0^t R(u)\,du$$

integration by parts yields that

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} R_1(t)\,t\,\frac{\partial}{\partial t}\left(\frac{\sin nt}{nt} - \cos nt\right)dt + 2R_1(\pi)\cos n\pi$$

$$= \frac{2}{\pi}R_1(\pi)\,h(\pi, c)\int_0^{\pi}\frac{u}{h(u,c)}\,\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu} - \cos nu\right)du$$

$$- \int_0^{\pi} d\{R_1(t)\,h(t,c)\}\int_0^t \frac{u}{h(u,c)}\,\frac{\partial}{\partial u}\left(\frac{\sin nu}{nu} - \cos nu\right)du$$

$$+ 2R_1(\pi)\cos n\pi$$

$$= O\left\{\frac{1}{n\,h(1/(n+2), c),}\right\}$$

$$+ \frac{2}{\pi}\int_0^{\pi} d\{R_1(t)\,h(t,c)\}\left[\frac{t\cos nt}{h(t,c)} - \frac{2\sin nt}{n\,h(t,c)}\right.$$

$$\left. - \frac{c\sin nt}{nh(t, c+1)\log(1/t)}\right]$$

since

$$\int_0^t \frac{u}{h(u,c)} \frac{\partial}{\partial u}\left(\frac{\sin nu}{nu} - \cos nu\right) du$$

$$= \frac{t}{h(t,c)}\left(\frac{\sin nt}{nt} - \cos nt\right) - \int_0^t \left(\frac{\sin nu}{nu} - \cos nu\right) \frac{d}{du}\left(\frac{u}{h(u,c)}\right) du$$

$$= -\frac{t\cos nt}{h(t,c)} + \frac{c\sin nt}{nh(t,c+1)\log(1/t)} + \frac{2\sin nt}{nh(t,c)}$$

$$-\int_0^t \frac{\sin nu}{nu}\left[\frac{1}{h(u,c)} + \frac{2c\left(\log\frac{1}{u}\right)^{-1} + c\left(\log\frac{1}{u}\right)^{-2}}{h(u,c+1)}\right.$$

$$\left. + \frac{c(c+1)\left(\log\frac{1}{u}\right)^{-2}}{h(u,c+2)}\right] du$$

$$= -\frac{t\cos nt}{h(t,c)} + \frac{c\sin nt}{n\,h(t,c+1)\log(1/t)} + \frac{2\sin nt}{n\,h(t,c)}$$

$$+ O\left\{\frac{1}{n\,h(1/(n+2),c)}\right\}$$

by Lemma 1, and since, by the hypothesis

$$\int_0^\pi |\, d\{R_1(t)\, h(t,c)\}\,|$$

is finite.

However, the method $|\,(C,1)(N,p_n)\,|$ is absolutely regular and hence

$$\sum \frac{y_n}{n\,h(1/(n+2),c)} \in |\,(C,1)(N,p_n)\,|\,.$$

Also, proceeding as in Theorem 1,

$$\Sigma\, \beta_n y_n \in |\,N,p_n\,|$$

and hence, a fortiori, summable $|\,(C,1)(N,p_n)\,|$, where

$$\beta_n = \frac{2}{n\pi}\int_0^\pi \left\{\frac{\sin nt}{h(t,c)} + \frac{c\sin nt}{h(t,c+1)\log(1/t)}\right\} d\{R_1(t)\, h(t,c)\}.$$

Thus for the proof, it would be sufficient to show that

$$\sum = \sum_{n=1}^{\infty} \Delta\left(\frac{1}{n}\right) \left| \sum_{m=1}^{n} \frac{m}{P_m P_{m-1}} \sum_{k=0}^{m-1} (P_m p_k - p_m P_k)\, y_{m-k} (\cos m-k)\, t \right|$$
$$= O(t^{-1})$$

uniformly in $0 < t < \pi$.

Now,

$$\sum = \sum_{n=1}^{\infty} \Delta\left(\frac{1}{n}\right) \left| \sum_{m=1}^{n} m \sum_{k=1}^{m} g(m, k)\, y_k \cos kt \right|$$

$$= \left(\sum_{n=1}^{2T+1} + \sum_{2T+2}^{\infty} \right) \left(\Delta\left(\frac{1}{n}\right) \left| \sum_{m=1}^{n} m \sum_{k=1}^{m} g(m, k)\, y_k \cos kt \right| \right)$$

$$\leqslant \Sigma_1 + \Sigma_2, \text{ say.}$$

By changing the order of summation and using the fact that $|\cos kt| \leqslant 1$ and $g(m, k) \geqslant 0$, we get

$$\sum\nolimits_1 \leqslant \sum_{m=1}^{2T+1} m \sum_{k=1}^{m} g(m, k)\, y_k \sum_{n=m}^{\infty} \Delta\left(\frac{1}{n}\right)$$

$$= \sum_{m=1}^{2T+1} \sum_{k=1}^{m} g(m, k)\, y_k$$

$$= \sum_{k=1}^{2T+1} y_k \sum_{m=k}^{2T+1} g(m, k)$$

$$\leqslant \sum_{k=1}^{2T} (y_k) \qquad \text{(by (3.1))}$$

$$= O(t^{-1})$$

uniformly in $0 < t < \pi$. And letting

$$x_k = \exp(ikt)$$

we have

$$\sum\nolimits_2 \leqslant \sum_{n=2T+2}^{\infty} \Delta\left(\frac{1}{n}\right)\left|\sum_{m=1}^{n}\frac{1}{P_m P_{m-1}}\sum_{k=0}^{m-1}(kP_m p_k(-mp_m P_k))\, y_{m-k}(x_{m-k})\right|$$

$$+\sum_{n=2T+2}^{\infty}\Delta\left(\frac{1}{n}\right)\left|\sum_{m=1}^{n}\frac{1}{P_{m-1}}\sum_{k=0}^{m}(m-k\ p_k)\, y_{m-k}(x_{m-k})\right|$$

$$= \Sigma_{2,1} + \Sigma_{2,2}, \text{ say.}$$

However

$$\sum\nolimits_{2,1} \leqslant \sum_{n=1}^{\infty}\Delta\left(\frac{1}{n}\right)\left|\sum_{m=1}^{n}\frac{1}{P_m P_{m-1}}\sum_{k=0}^{m-1}(kP_m p_k(-mp_m P_k))\, y_{m-k}(x_{m-k})\right|$$

$$= \sum_{n=1}^{\infty}\Delta\left(\frac{1}{n}\right)\left|\sum_{m=1}^{n}\frac{1}{P_{m-1}}\sum_{k=0}^{m-1}(C_k)-C_{mk}(P_k)\, y_{m-k}(x_{m-k})\right|$$

$$= \sum_{n=1}^{\infty}\Delta\left(\frac{1}{n}\right)\left|\sum_{m=1}^{n}\frac{1}{P_{m-1}}\sum_{k=0}^{m-1}P_k\, y_{m-k}(x_{m-k})\sum_{\mu=k}^{m-1}\Delta C_\mu\right|$$

$$= \sum_{n=1}^{\infty}\Delta\left(\frac{1}{n}\right)\sum_{m=1}^{n}\frac{1}{P_{m-1}}\sum_{\mu=0}^{m-1}\Delta C_\mu\sum_{k=0}^{\mu}P_k\, y_{m-k}(x_{m-k})\Big|$$

$$= O(t^{-1})\sum_{n=1}^{\infty}\Delta\left(\frac{1}{n}\right)\sum_{m=1}^{n}\frac{1}{P_{m-1}}\sum_{\mu=0}^{m-1}|\Delta C_\mu|\, P_\mu\, y_{m-\mu}$$

(by Abel's lemma)

$$= O(t^{-1})\sum_{m=1}^{\infty}\frac{1}{m P_{m-1}}\sum_{\mu=0}^{m-1}|\Delta C_\mu|\, P_\mu\, y_{m-\mu}$$

$$= O(t^{-1})\sum_{\mu=0}^{\infty}|\Delta C_\mu|\, P_\mu\sum_{m=\mu+1}^{\infty}\frac{y_{m-\mu}}{m P_{m-1}}$$

$$= O(t^{-1})\sum_{\mu=0}^{\infty}|\Delta C_\mu|$$

$$= O(t^{-1})$$

by (2.7) and since

$$\sum_{m-\mu+1}^{\infty} \frac{y_{m-\mu}}{m\,P_{m-1}} = \sum_{m-\mu+1}^{2\mu} \frac{y_{m-\mu}}{m\,P_{m-1}} + \sum_{m=2\mu+1}^{\infty} \frac{y_{m-\mu}}{m\,P_{m-1}}$$

$$= O\left(\frac{y_\mu}{P_\mu}\right) + O(1) \sum_{m=\mu}^{\infty} \frac{y_{m-\mu}}{m\,P_m}$$

$$= O(1/P_\mu)$$

by (2.8).

Finally, we consider

$$\sum\nolimits_{2,2} = \sum_{n=2T+2}^{\infty} \frac{1}{n(n+1)} \left| \sum_{m=0}^{n-1} \frac{1}{P_m} \sum_{k=0}^{m} (m+1-k)\, p_k\, y_{m-k+1}\, x_{m+1-k} \right|$$

$$= \sum_{n=2T+2} \frac{1}{n(n+1)} \left| \sum_{k=0}^{n-1} p_k \sum_{m=k}^{n-1} \frac{(m+1-k)\, y_{m-k+1}\, x_{m+1-k}}{P_m} \right|$$

where

$$\sum_{m=k}^{n-1} \frac{(m+1-k)\, y_{m-k+1}}{P_m}\, x_{m+1-k}$$

$$= \frac{x_1}{1-x_1} \left[\frac{y_1}{P_k} - \frac{(n+1-k)\, y_{n+1-k}\, x_{n-k}}{P_n} \right.$$

$$\left. - \sum_{m=k}^{n-1} x_{m+1-k}\, \Delta_m \left(\frac{(m+1-k)\, y_{m-k+1}}{P_m} \right) \right]$$

$$= \frac{x_1}{1-x_1} \left[\frac{y_1}{P_k} - \frac{(n+1-k)\, y_{n+1-k}\, x_{n-k}}{P_n} \right.$$

$$- \sum_{m=k}^{n-1} \frac{p_{m+1}}{P_m P_{m+1}} (m+1-k)\, y_{m+1-k}\, x_{m+1-k}$$

$$\left. + \sum_{m=k}^{n-1} \frac{y_{m+1-k}}{P_{m+1}}\, x_{m+1-k} - \sum_{m=k}^{n-1} \frac{(m+2-k)\, \Delta_m\, y_{m+1-k}}{P_{m+1}}\, x_{m+1-k} \right]$$

and

$$\frac{(m+2-k)\, \Delta_m\, y_{m+1-k}}{P_{m+1}}\, x_{m+1-k} = \frac{y_{m+2-k}\, y_{m+2-k}\, x_{m+1-k}}{P_{m+1}}$$

$$+ O\left\{ \frac{y^2_{m-k+1}}{(m+1-k)\, P_{m+1}} \right\}.$$

Therefore

$$\sum_{m=k}^{n-1}\frac{(m+1-k)\,y_{m+1-k}\,x_{m+1-k}}{P_m}=O\left(\frac{1}{t\,P_k}\right)$$

$$-\frac{x_1}{1-x_1}\left[\frac{(n+1-k)\,y_{n+1-k}\,x_{n-k}}{P_n}\right.$$

$$+\sum_{m=k}^{n-1}\frac{(m+1-k)\,p_{m+1}}{P_m\,P_{m+1}}\,y_{m+1-k}\,x_{m+1-k}$$

$$+\sum_{m=k}^{n-1}\frac{y_{m+1-k}\,y_{m+2-k}\,x_{m+1-k}}{P_{m+1}}$$

$$\left.-\sum_{m=k}^{n-1}\frac{y_{m+1\ k}\,x_{m+1-k}}{P_{m+1}}\right]. \qquad \ldots(5.1.1)$$

Now, the contribution of the first term of (5.1.1) to $\Sigma_{2,2}$ does not exceed

$$O(t^{-1})\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=1}^{n}\frac{p_k}{P_k}$$

$$=O(t^{-1})\sum_{k=1}^{\infty}\frac{p_k}{k\,P_k}$$

$$=O(t^{-1})$$

uniformly in $0<t<\pi$, and the contribution of the second term of (5.1.1) to $\Sigma_{2,2}$ does not exceed

$$O(t^{-1})\sum_{n=2T+2}^{\infty}\frac{1}{n(n+1)P_n}\left|\sum_{k=0}^{n-1}(n+1-k)\,y_{n+1-k}\,p_k\,x_{n-k}\right|$$

$$=O(t^{-1})\sum_{n=2T+2}^{\infty}\frac{y_{n+1}}{n\,P_n}\max_{0\leq k'\leq n-1}\left|\sum_{k=k'}^{n-1}p_k\,x_{n-k}\right|$$

$$=O(t^{-1})\,P_T\sum_{n=T}^{\infty}\frac{y_{n+1}}{n\,P_n}$$

$$=O(t^{-1})$$

by Abel's lemma, Lemma 3 and (2.8). Contribution of the third term of (5.1.1) to $\Sigma_{2,2}$ does not exceed

$$O(t^{-1}) \sum_{n=2T+2}^{\infty} \Delta\left(\frac{1}{n}\right) \left| \sum_{k=0}^{n-1} p_k \sum_{m=k}^{n-1} \frac{(m+1-k)\, p_{m+1}\, y_{m+1-k}}{P_m\, P_{m+1}}\, x_{m+1-k} \right|$$

$$= O(t^{-1}) \sum_{n=2T+2}^{\infty} \Delta\left(\frac{1}{n}\right) \left| \sum_{m=0}^{n-1} \Delta\left(\frac{1}{P_m}\right) \sum_{k=0}^{m} (m+1-k)\, p_k\, y_{m+1-k}\, x_{m+1-k} \right|$$

$$= S(t), \text{ suppose.}$$

Now,

$$\left| \sum_{m=0}^{n-1} \Delta\left(\frac{1}{P_m}\right) \sum_{k=0}^{m} (m+1-k)\, p_k\, y_{m+1-k}\, x_{m+1-k} \right|$$

$$\leqslant y_1 \sum_{m=0}^{T-1} \Delta\left(\frac{1}{P_m}\right) \sum_{k=0}^{m} (m+1-k)\, p_k$$

$$+ \sum_{m=T}^{n-1} \Delta\left(\frac{1}{P_m}\right) \left| \sum_{k=0}^{m} (m+1-k)\, p_k\, y_{m+1-k}\, x_{m+1-k} \right|$$

$$= O(t^{-1}) + O(P_T) \sum_{m=T}^{n-1} (m+1)\, \Delta\left(\frac{1}{P_m}\right)$$

so that, we have

$$S(t) = O(t^{-1}) + O(t^{-1} P_T) \sum_{n=T+1}^{\infty} \Delta\left(\frac{1}{n}\right) \sum_{m=T+1}^{n-1} (m+1)\, \Delta\left(\frac{1}{P_m}\right)$$

$$= O(t^{-1}) + O(t^{-1} P_T) \sum_{m=T}^{\infty} (m+1)\, \Delta\left(\frac{1}{P_m}\right) \sum_{n=m+1}^{\infty} \Delta\left(\frac{1}{n}\right)$$

$$= O(t^{-1}).$$

The contributions of the fourth term and the last term of (5.1.1) to $\Sigma_{2,2}$ would be estimated in a similar way.

We consider the contribution of the last term; it is clear that the contribution of the fourth term can be dealt with in a similar way. The contribution of the last term to $\Sigma_{2,2}$ does not exceed

$$O(t^{-1}) \sum_{n=2T+2}^{\infty} \Delta\left(\frac{1}{n}\right) \left| \sum_{k=0}^{n-1} p_k \sum_{m=k}^{n-1} \frac{y_{m+1-k}\, x_{m+1-k}}{P_{m+1}} \right|$$

$$= O(t^{-1}) \sum_{n=T+1}^{\infty} \Delta\left(\frac{1}{n}\right) \left| \sum_{m=0}^{n-1} \frac{1}{P_{m+1}} \sum_{k=0}^{m} p_k\, y_{m+1-k}\, x_{m+1-k} \right|$$

where, by Lemma 7,

$$\left| \sum_{k=0}^{m} p_k\, y_{m+1-k}\, x_{m+1-k} \right| = \left| \sum_{k=1}^{m+1} p_{m+1-k}\, y_k\, x_k \right|$$

$$= O\{t^{-1} p_N\, y_T\} + O\{P_T\, y_{m+1}\}$$

where $N = [\frac{1}{2}(m+1)]$, the integral part of $\frac{1}{2}(m+1)$. Thus

$$\sum_{m=T+1}^{\infty} \Delta\left(\frac{1}{n}\right) \left| \sum_{m=0}^{n-1} \frac{1}{P_{m+1}} \sum_{k=0}^{m} p_k\, y_{m+1-k}\, x_{m+1-k} \right|$$

$$\leqslant \sum_{n=T}^{\infty} \Delta\left(\frac{1}{n}\right) \sum_{m=0}^{T-1} \frac{P_m}{P_{m+1}} + \sum_{n=T}^{\infty} \Delta\left(\frac{1}{n}\right) \sum_{m=T}^{n-1} \frac{1}{P_{m-1}} \left| \sum_{k=0}^{m-1} p_k y_{m+1-k} x_{m+1-k} \right|$$

$$= O(1) + O(t^{-1} y_T) \sum_{n=T}^{\infty} \Delta\left(\frac{1}{n}\right) \sum_{m=T}^{n} \frac{p_N}{P_{m+1}}$$

$$+ O(P_T) \sum_{n=T}^{\infty} \Delta\left(\frac{1}{n}\right) \sum_{m=T}^{n} \frac{y_{m+1}}{P_m}$$

$$= O(1) + O(t^{-1}) \sum_{m=T}^{\infty} \frac{p_N}{m P_{m+1}} + O(P_T) \sum_{m=T}^{\infty} \frac{y_{m+1}}{m P_m}$$

$$= O(1) + O(t^{-1}) \sum_{m=T}^{\infty} \Delta\left(\frac{1}{m}\right) + O(1)$$

$$= O(1)$$

by (2.8) and the fact that

$$p_N = O(P_N/m).$$

Thus the contribution of the last term of (5.1.1) to $\Sigma_{2,2}$ does note exceed $O(t^{-1})$. Now combining the results, as obtained above, we get

$$\Sigma_{2,2} = O(t^{-1}),$$

uniformly in $0 < t < \pi$.

This completes the proof.

5.2. *The condition is necessary*—We have

$$A_n(x) = 2R_1(\pi)\cos n\pi + \frac{2}{\pi}\int_0^{\pi} R_1(t)\, t\, \frac{\partial}{\partial t}\left(\frac{\sin nt}{nt} - \cos nt\right) dt.$$

Now, we choose

$$R(t) = \frac{d}{dt}\,(t/h(t, c))$$

which is possible, since $R(t) \in L(0, \pi)$ and hence by Lemma 6 there exists some $\phi(t) = \in L(0, \pi)$. Then

$$R_1(t) = (1/t)\int_0^t R(u)\, du = 1/h(t, c)$$

and

$$A_n(x) = \frac{2\cos n\pi}{h(\pi, c)} + \frac{2}{\pi}\Bigg[-\frac{\pi\cos n\pi}{h(\pi, c)} - \int_0^{\pi}\frac{\sin nt}{nt}\Bigg\{\frac{1}{h(t, c)} + \frac{2c\left(\log\frac{1}{t}\right)^{-1} + c\left(\log\frac{1}{t}\right)^{-2}}{h(t, c+1)} + \frac{c(c+1)\left(\log\frac{1}{t}\right)^{-2}}{h(t, c+2)}\Bigg\} dt\Bigg]$$

$$= -\frac{2}{n\pi}\int_0^{\pi}\frac{\sin nt}{t\,h(t, c)}\, dt - \frac{2c}{n\pi}\int_0^{\pi}\frac{\sin nt}{t}\left\{\frac{2\left(\log\frac{1}{t}\right)^{-1} + \left(\log\frac{1}{t}\right)^{-2}}{h(t, c+1)} + \frac{(c+1)\left(\log\frac{1}{t}\right)^{-2}}{h(t, c+2)}\right\} dt$$

$$= g_n + r_n, \text{ say.}$$

By Lemmas 1 and 2, respectively

$$r_n = O\left\{\frac{y_n}{n\,h(1/(n+2), c+1)}\right\}$$

and

$$g_n\, y_n \sim -\frac{y_n}{n\,h(1/(n+2), c)}.$$

Thus the necessity of (2.6) may be followed as in Theorem 1.

Also arguing as in Theorem 1 it may be observed that (y_n) cannot be replaced by (y_n^{β}) $(0 < \beta < 1)$ for any real c.

This completes the proof of the theorem.

6. COROLLARIES

We give a few results in the form of corollaries of Theorems 1 and 2.

Corollary 1—Let (2.1) hold and let (p_n) be non-negative and non-increasing such that (2.8) hold. Then $\Sigma A_n(x) y_n \in |N, p_n|$.

This improves Theorem *B* which is due to Kanno[6].

PROOF: We have

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} \phi(t) \cos nt \, dt$$

$$= -\frac{2}{\pi}\int_0^{\pi} \frac{\sin nt}{n} \, d\phi(t).$$

Since $\frac{2}{\pi}\int_0^{\pi} |d\phi(t)|$ is finite, the proof may be completed if we show that (see Theorem 1 of this paper)

$$\sum_{n=1}^{\infty} \frac{1}{P_n P_{n-1}} \left| \sum_{m=1}^{n} (P_n p_{n-m} - p_n P_{n-m}) \frac{y_m \sin mt}{m} \right| = O(1) \qquad \ldots(6.1)$$

uniformly in $0 < t < \pi$.

Now, proceeding as in Theorem 1 and using (2.8) in place of (2.4), the proof of (6.1) may be completed.

Corollary 2—Let $R(t) h(t, c) \in BV(0, \pi)$. Then $\Sigma A_n(x) y_n \in \left|N, \frac{1}{n+1}\right|$ if and only if $c > 1$. Further, (y_n) cannot be replaced by (y_n^{β}) $(0 < \beta < 1)$.

This is a direct consequence of Theorem 1 for $p_n = (n+1)^{-1}$.

Corollary 3—Let $\phi_1(t) \in BV(0, \pi)$ and let (p_n) be a non-negative non-increasing such that (2.7) and (2.8) hold. Then $\Sigma A_n(x) y_n \in |(C, 1)(N, p_n)|$.

PROOF: We have

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} \phi(t) \cos nt \, dt.$$

Then integrating by parts 2-times and using $\phi_1(\pi) = 0$, we get

$$A_n(x) = \frac{2}{\pi}\int_0^{\pi} t \cos nt \, d\phi_1(t) - q_n$$

where

$$q_n = \frac{2}{\pi}\int_0^{\pi} \frac{\sin nt}{n}\, d\,\phi_1(t).$$

Proceeding as in Theorem 1 and using (2.8) in place of (2.4), one may obtain that

$$\Sigma\, q_n y_n \in |N, p_n|$$

and hence, a fortiori, summable $|(C, 1)(N, p_n)|$. Thus to complete the proof of the corollary, it would be sufficient to prove that

$$\sum_{n=1}^{\infty} \Delta\left(\frac{1}{n}\right)\left|\sum_{m=1}^{n} \frac{m}{P_m P_{m-1}} \sum_{k=0}^{m-1} (P_m p_k - p_m P_k)\, y_{m-k} \cos(m-k)\, t\right| = O(t^{-1}) \qquad ...(6.2)$$

uniformly in $0 < t < \pi$, since $\int_0^{\pi} |d\,\phi_1(t)|$ is finite. And for the proof of (6.2), see the estimate of Σ in Theorem 2.

Carollary 4—Let (*) hold and let $R_1(t)\, h(t, c) \in BV(0, \pi)$. Then $\Sigma A_n(x)\, y_n \in \left|(C, 1)\left(N, \frac{1}{n+1}\right)\right|$ if and only if $c > 1$. Further, (y_n) cannot be replaced by (y_n^{β}) $(0 < \beta < 1)$.

This is a direct consequence of Theorem 2 for $p_n = (n+1)^{-1}$.

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REFERENCES

1. P. Chandra, *Monats. Math.* **77** (1973) 289–98.
2. P. Chandra. *Periodica Math. Hung.* **5** (1974) 121–30.
3. P. Chandra, *Indian J. pure appl. Math.* **9** (1978) 1004–18.
4. P. Chandra, *Jñānabha* **8** (1978) 17–24.
5. P. Chandra, *Indian J. pure. appl. Math.* **11** (1980) 215–29.
6. K. Kanno, *Tohoku Math. J.* **21** (1969) 434–47.
7. K. Knopp, and G. G. Lorentz, *Arch. der Math.* **2** (1949) 10–16.
8. S. N. Lal, *Publ. Inst. Math.* **9** (23) (1969) 13–28.
9. S. N. Lal, *Indian J. Math.* **16** (1974) 1–22.
10. L. McFadden, *Duke Math. J.* **9** (1942) 168–207.
11. F. M. Mears, *Bull. Am. Math. Soc.* **41** (1935) 875–80.
12. F. M. Mears, *Annals Math.* **38** (1937) 594–601.
13. N. E. Nörlund, *Lunds Universitets Årsskrift* (2) **16** (1919) No. 3
14. T. Singh, *Indian J. Math.* **9** (1967) 227–36.
15. O. P. Varshney, *Proc. Am. Math. Soc.* **10** (1959)784–89.
16. A. Zygmund, *Trigonometric Series* Vol. I. Cambridge University Press (1959).