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CONFLUENT HYPERGEOMETRIC FUNCTION OF SECOND KIND WITH MATRIX ARGUMENT

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In this paper, we have generalized confluent hypergeometric function of second kind to the matrix argument. Its connections with certain other special functions defined by Herz, some of its differential properties, and an interesting confluence have also been established. In the last section, we have applied our theory in the field of multivariate statistical analysis by finding the distribution of Bartlett's criterion :

$$V_1 = \frac{(\det A_1)^{\frac{1}{2}n_1} (\det A_2)^{\frac{1}{2}n_2}}{\det (A_1 + A_2)^{\frac{1}{2}(n_1+n_2)}}$$

for testing the hypothesis : $H_1 = \Sigma_1 = \Sigma_2$.

1. INTRODUCTION

A function of matrix argument is a real or complex valued function of the elements of a matrix. Herz[6] has defined a Laplace transform of the functions of matrix argument in the space of $m \times m$ real symmetric matrices: G_m. He has also mentioned many properties of this transform which are analogous to the Laplace transform with one variable. With the help of this transform he has developed hypergeometric functions of matrix argument. These functions also appear implicitly in the works of Anderson[1], James[7], and Constantine[4], who are workers in the field of multivariate statistical analysis. Herz has also given a general definition of the functions of second kind, *cf.* Herz[6] (*loc. cit.*, 5.1), but he has discussed only Bessel function of second kind, $B_\delta(Z)$, in some detail.

In the present paper, we have developed confluent hypergeometric function of second kind with matrix argument, $\psi(a; c; Z)$. Section 2 contains the definitions. In section 3, we have proved some integral formulae relating $\psi(a; c; Z)$ with certain other special functions of matrix argument defined by Herz. Section 4 contains some differential properties. In section 5, some special cases of $\psi(a; c; Z)$ have been found and in section 6, a confluence of $\psi(a; c; Z)$ with Bessel function of second kind $B_\delta(Z)$, has been established. We remark that the theory developed in the present paper finds application in multivariate statistical analysis. In section 7, we have given few examples in support of our remark.

Throughout this paper, capital letters will denote $m\times m$ symmetric matrices. If Λ is positive definite, we shall write $\Lambda > 0$. etr (Λ) and det (Λ) will stand for $e^{\text{trace}(\Lambda)}$ and determinant (Λ). E will be $m\times m$ unit matrix, and $\Gamma_m(a) = \pi^{\frac{m(m-1)}{4}} \cdot \prod_{i=1}^{m} \Gamma\left(a-\frac{i-1}{2}\right)$.

We feel it necessary to mention a Jacobian of transformation in the space G_m. For the transformation $\Lambda \rightarrow T\Lambda T' = M$, where $\Lambda, M \in G_m$, the Jacobian of transformation is equal to $(\det T)^{2p}$; $p = (m+1)/2$, if the measure in G_m is defined by $d\Lambda = \prod_{i\leqslant j} d\lambda_{ij}$ (*cf.* Olkin[9]).

2. Definitions

One of the wellknown integral representation of confluent hypergeometric function is

$$\psi(a; c; x) = \frac{1}{\Gamma(a)} \cdot \int_0^\infty e^{-xt}\, t^{a-1} (1+t)^{c-a-1}\, dt. \qquad ...(1)$$

In analogy with (1), we define confluent hypergeometric function of second kind by the following equation

$$\psi(a, c; Z) = \frac{1}{\Gamma_m(a)} \int_{\Lambda>0} \operatorname{etr}(-\Lambda Z)(\det \Lambda)^{a-p} \det(E+\Lambda)^{c-a-p}\, d\Lambda \quad ...(2)$$

where $d\Lambda = \prod_{i\leqslant j} d\lambda_{ij}$, and the integral is taken over the set of positive definite matrices. The function defined by (2) is a Laplace transform provided the integral (2) converges. The condition of convergence of (2) are

$$\text{Re}(Z) > 0, \text{ and Re}(a) > p-1.$$

Thus (2) defines a function in the generalized right half plane Re $(Z) > 0$, provided Re $(a) > p-1$.

It is very easy to establish the absolute convergence of integral (2) under the conditions stated. Because det $(E+\Lambda)^k$ is of exponential order for Re $(k) > 0$, and bounded for Re $(k) \leqslant 0$.

Now, changing variables in (2) from Λ to $Z^{-1/2}\Lambda Z^{-1/2}$ where Z is real symmetric, we have

$$\psi(a, c; Z)(\det Z)^{a} = \frac{1}{\Gamma_m(a)} \int_{\Lambda>0} \operatorname{etr}(-\Lambda)(\det \Lambda)^{a-p} \det(E+M^{-1}\Lambda)^{c-a-p}\, d\Lambda \quad \ldots(3)$$

or

$$\psi(a, c; Z)(\det Z)^{c-p} = \frac{1}{\Gamma_m(a)} \int_{\Lambda>0} \operatorname{etr}(-\Lambda)(\det \Lambda)^{a-p} \det(\Lambda+Z)^{c-a-p}\, d\Lambda. \quad \ldots(4)$$

Equations (3) and (4) hold for Re $(Z) > 0$ provided Re $(a) > p-1$. As a consequence of (2) and (4) we have

$$\psi(a, c; MZ)(\det M)^{a} = \frac{1}{\Gamma_m(a)} \int_{\Lambda>0} \operatorname{etr}(-\Lambda Z)(\det \Lambda)^{a-p} \det \times (E+\Lambda M^{-1})^{c-a-p}\, d\Lambda \quad \ldots(5)$$

holding for Re (M), Re $(Z) > 0$, and Re $(a) > p-1$. Equation (5) defines ψ function for all values of argument except for those values for which real part is non positive definite and imaginary part is null matrix. We note that Re $(a) > 0$ is required every where in the development of the definition of ψ function. We have not been able to remove this restriction.

3. Some Integral Formulae

First of all we will give an integral representation of $\psi(a, c; Z)$ as a Laplace transform of a function involving Bessel function of second kind, $B_\delta(Z)$. This function has been defined for matrix variables by Herz[6]. He has also shown that

$$B_\delta(WZ)(\det W)^{\delta} = \int_{\Lambda>0} \operatorname{etr}(-\Lambda W) \operatorname{etr}(-\Lambda^{-1} Z)(\det \Lambda)^{-\delta-p}\, d\Lambda \quad \ldots(6)$$

holding for all δ provided Re (W), Re $(Z) > 0$. Herz has also shown that

$$B_{-\delta}(Z) = B_\delta(Z)(\det Z)^{\delta-p}. \quad \ldots(7)$$

To prove our result, we consider the Laplace transform of $(\det M)^{a-\delta-p} B_{-\delta}(M)$. Taking Laplace transform of $(\det M)^{a-\delta-p} B_{-\delta}(M)$, applying (6), and changing the order of integration, we have

$$\int_{M>0} \operatorname{etr}(-MZ)(\det M)^{a-\delta-p} B_{-\delta}(M)\, dM$$

$$= \int_{\Lambda>0} \operatorname{etr}(-\Lambda)(\det \Lambda)^{\delta-p} \left\{ \int_{M>0} \operatorname{etr}(-M(Z+\Lambda^{-1}))(\det M)^{a-\delta-p}\, dM \right\} d\Lambda.$$

Now, from eqns. (1.1) and (2.4) of Herz[6], we have

$$\int_{M>0} \operatorname{etr}(-MZ)(\det M)^{a-\delta-p} B_{-\delta}(M)\, dM = \Gamma_m(a)\,\Gamma_m(a-\delta)(\det Z)^{-a} \times \psi(a, \delta+p; Z^{-1}) \quad \ldots(8)$$

holding for **Re** $(a) > p-1$ and Re $(a-\delta)\ p-1$, and Re $(Z) > 0$. Applying (7) in (8), we have

$$\int_{M>0} \operatorname{etr}(-MZ)(\det M)^{a-p})\, B_\delta(M)\, dM = \Gamma_m(a)\,\Gamma_m(a-\delta)(\det Z)^{-a}$$

$$\times\ \psi(a, \delta+p; Z^{-1}) \qquad \ldots(9)$$

holding under the conditions of (8).

Now, we will generalize the classical Meijer's integral representation of confluent hypergeometric function. Let L^2_γ be the Hilbert space of functions of matrix argument for which the norm is defined by

$$\|f\|^2_\gamma = \int_{R>0} |f(R)|^2 (\det R)^\gamma\, dR < \infty.$$

Herz (loc. cit.) has established the identity :

$$\int_{R>0} \operatorname{etr}(-RZ)\, f(R)(\det \mathrm{R})^\gamma\ dR = (\det\ Z)^{-\gamma-p} \int_{\Lambda>0} \operatorname{etr}(-\Lambda Z^{-1})\ g(\Lambda)$$

$$(\det(\Lambda)^\gamma\, d\Lambda \qquad \ldots(10)$$

where f and g belong to L^2_γ, and g is γ—Hankel transform of f. For the definition of Hankel transform (*cf.* Herz[6]). Herz has also shown that

$$\int_{\Lambda>0} B_{\alpha-\beta}(\Lambda)\, A_{\gamma-p}(\Lambda R)(\det\Lambda)^{\alpha-p}\, d\Lambda = \frac{\Gamma_m(\alpha)\,\Gamma_m(\beta)}{\Gamma_m(\gamma)}\ {}_2F_1(\alpha, \beta, \gamma; -R) \qquad \ldots(11)$$

holding for Re $(Z) > 0$, Re (α), Re $(\beta) > p-1$, where $A_{\gamma-p}$ and ${}_2F_1$ are Bessel and hypergeometric functions of matrix argument respectively.

Now, in view of (11) $(\gamma-p)$-Hankel transform of $(\det\Lambda)^{\alpha-\gamma} B_{\alpha-\beta}(\Lambda)$ is

$$\frac{\Gamma_m(\alpha)\,\Gamma_m(\beta)}{\Gamma_m(\gamma)}\ {}_2F_1(\alpha, \beta; \gamma; -R).$$

So from (10), we have

$$\int_{R>0} \operatorname{etr}(-RZ)(\det R)^{\alpha-\gamma} B_{\alpha-\beta}(R)(\det R)^{\gamma-p}\, dR$$

$$= \frac{\Gamma_m(\gamma)\,\Gamma_m(\beta)}{\Gamma_m(\gamma)} (\det Z)^{-\gamma} \int_{\Lambda>0} \operatorname{etr}(-\Lambda Z^{-1})\, {}_2F_1(\alpha, \beta, \gamma; -\Lambda)(\det\Lambda)^{\gamma-p}.$$

Now, from (9) we obtain

$$\int_{\Lambda>0} \operatorname{etr}(-\Lambda Z^{-1})\, {}_2F_1(\alpha, \beta; \gamma; -\Lambda)(\det\Lambda)^{\rho-p} d\Lambda$$

$$= \Gamma_m(\gamma)(\det Z)^{-(\alpha-\gamma)}\, \psi(\alpha, \alpha-\beta+p; Z^{-1})$$

Writing Z in place of Z^{-1}, we have Meijer integral in the form

$$\int\limits_{\Lambda>0} \operatorname{etr}(-\Lambda Z)\, {}_2F_1(\alpha, \beta; \gamma; -\Lambda)\, (\det \Lambda)^{\gamma-p}\, d\Lambda$$

$$= \Gamma_m(\gamma)\, (\det Z)^{(\alpha-\gamma)}\, \psi(\alpha, \alpha-\beta+p; Z) \qquad \ldots(12)$$

holding for Re (α), Re (β), Re $(\gamma) > p-1$, and Re $(Z) > 0$.

Herz[6] has shown that in the simply connected region : Re (α), Re (β), Re $(\gamma) > p-1$, ${}_2F_1(\alpha, \beta; \gamma; Z) = {}_2F_1(\beta, \alpha; \gamma; Z)$. So interchanging α and β in (7) for sufficiently large α and β, we get

$$\psi(\alpha, \alpha-\beta+p; Z) = (\det Z)^{\beta-\alpha}\, \psi(\beta, \beta-\alpha+p; Z),$$

holding for Re (α), Re $(\beta) > p-1$. Now, writing a in place of β, and c in place of $\beta-\alpha+p$, we obtain

$$(\det Z)^{c-p}\, \psi(a, c; Z) = \psi(a-c+p, 2p-c, Z) \qquad \ldots(13)$$

provided Re (a), Re $(a-c+p) > p-1$.

4. Some Differential Properties

Garding[5] has defined operator of fractional integration and hyperbolic differential operator. The definition of the differential operator is as follows.

$D_Z = \det\left(\dfrac{\partial}{\partial z_{ij}}\right)$, provided Z is parametrized as $(\eta_{ij}\, z_{ij})$; η_{ij} being 1 if $i = j$, and 1/2 otherwise.

$D_\Lambda = \det(\eta_{ij}\, \lambda_{ij})$; η_{ij} being 1 if $i = j$ and 1/2 otherwise, provided Λ is parametrized as (λ_{ij}). Herz (loc. cit.) has shown that

$$D_Z \operatorname{etr}(\Lambda Z) = \det(\Lambda) \operatorname{etr}(\Lambda Z),$$

and

$$D_\Lambda \operatorname{etr}(\Lambda Z) = \det(Z) \operatorname{etr}(\Lambda Z).$$

We shall apply this differential operator to deduce the differential properties.

Applying D operator k times under the sign of integration in (2) we have

$$D_Z^k\, \psi(a, c; Z) = (-1)^{mk}. \frac{\Gamma_m(a+k)}{\Gamma_m(a)}\, \psi(a+k, c+k; Z) \qquad \ldots(14)$$

provided Re $(a) > p-1$, and Re $(Z) > 0$. D-differentiation is permissible under the sign of integration since the integral (2) is an absolutely convergent Laplace integral.

Similarly, applying D-operator k-times under the sign of integration in (9) we have

$$D_Z^k \ (\det Z)^{-a} \psi (a, \delta + p; Z^{-1})$$

$$(-1)^{mk}. \frac{\Gamma_m (a+k) \Gamma_m (a+k-\delta)}{\Gamma_m (a) \Gamma_m (a-\delta)} . \psi (a+k, \delta+p; Z^{-1}) (\det Z)^{-a-k}. \quad \ldots(15)$$

Now, applying D-operator k-times under the sign of integration in (4) we have

$$D_M^k \ \psi (a, c; M) (\det (M)^{a-p}$$

$$= \frac{1}{\Gamma_m (a)} \int_{\Lambda > 0} \operatorname{etr} (-\Lambda) (\det \Lambda)^{a-p} \ C_M^k \ [\det (\Lambda + M)^{c-a-p}] \, d\Lambda$$

But, from Lemma 9.2 of Garding[5], we know that

$$D_M^k \ \det (\Lambda + M)^{c-a-p} = \frac{\Gamma_m (c-a)}{\Gamma_m (c-k-a)} . \det (\Lambda - M)^{c-a-p-k}$$

Therefore

$$D_M^k \ \psi (a, c; M) (\det M)^{a-p} = \frac{\Gamma_m (c-a)}{\Gamma_m (c-k-a)} \psi (a, c-k; M) (\det M)^{c-k-p}. \quad \ldots(16)$$

5. Some Special Cases

We shall now derive a few special cases of ψ function. Following result is a generalization of exponential integral to the matrix argument.

$$\int_{\Lambda > R} \operatorname{etr} (-\Lambda) (\det \Lambda)^{-p} \, d\Lambda = \Gamma_m (p) \operatorname{etr} (-R) \psi (p, p; R); \ R > 0. \quad \ldots(17)$$

Here the region of integration; denoted bp $\Lambda > R$, is actually the set of all *p.d.* Λ for which $\Lambda - R$ is also *p.d.*. This is a radiated cone with vertex at R.

To prove (17), we make a change of variables from Λ to $\Lambda + R$. Under this change, the Jacobian of transformation remains unchanged, and the region of integration now becomes the set af all *p.d.* matrices. So we have

$$\int_{\Lambda > R} \operatorname{etr} (-\Lambda) (\det \Lambda)^{-p} \, d\Lambda = \operatorname{etr} (-R) \int_{\Lambda > 0} \operatorname{etr} (-\Lambda) \det (\Lambda + R)^{-p} \, d\Lambda.$$

Now, from (4) we have

$$\int_{\Lambda > R} \operatorname{etr} (-\Lambda) (\det \Lambda)^{-p} \, d\Lambda = \Gamma_m (p) \operatorname{etr} (-R) \psi (p, p; R).$$

Another special case of ψ function is the incomplete gamma function. We shall now establish this fact for the case of functions of matrix argument. We show that

$$\int_{\Lambda>R} \text{etr}(-\Lambda)(\det(\Lambda)^{a-p}\, d\Lambda = \Gamma_m(p)\,\text{etr}(-R)(\det R)^a\, \psi(p, a+p; R). \quad \text{...(18)}$$

To prove the result, we make same change of variables as in proving (17). Consequently, we have the result. In the particular case **Re** $(a) < 1$, we can apply (13) in (18) to get

$$\int_{\Lambda>R} \text{etr}(-\Lambda)(\det \Lambda)^{a-p}\, d\Lambda = \Gamma_m(p)\,\text{etr}(-R)\, \psi(p-a, p-a; R). \quad \text{...(19)}$$

6. A Confluence

We shall now establish the confluence :

$$\lim_{a\to\infty} \quad \Gamma_m(a-c-p)\, \psi(a, c; \frac{1}{a}\Lambda) = B_{c-p}(\Lambda). \quad \text{...(20)}$$

The author[8] has proved that if $f(Z^{1/2}\Lambda Z^{1/2})$ processes continuous first order partial derivatives in G_m for $\Lambda > 0$, then

$$\lim_{K\to\infty} \frac{k^{(k+p)m}}{\Gamma_m(k+p)} (\det Z)^{-k-p} \int_{\Lambda>0} \text{etr}(-kZ^{-1}\Lambda)(\det \Lambda)^k f(\Lambda)\, d\Lambda$$

$$= f(Z);\ Z > 0 \quad \text{...(21)}$$

provided

$$\int_{\Lambda>0} \text{etr}(-k\Lambda) f(Z^{1/2}\Lambda Z^{1/2})\, d\Lambda < \infty;\ k > 0,\ Z > 0.$$

We shall use (21) in proving the required confluence. From (9), we have by writing aZ^{-1} in place of Z,

$$\int_{M>0} \text{etr}(-aZ^{-1}M)(\det M)^{a-p} B_\delta(M)\, dM$$

$$= (a)^{-ma}\, \Gamma_m(a)\, \Gamma_m(a-\delta)\, (\det Z)^a\, \psi(a, \delta+p; \frac{1}{a} Z).$$

Now, applying (21), we have

$$\lim_{a\to\infty} \frac{\Gamma_m(a)\, a^{pm}}{\Gamma_m(a+p)} \Gamma_m(a-\delta)\, \psi(a, \delta+p; \frac{1}{a} Z) = B_\delta(Z).$$

Writing $\delta + p = c$, and noting that

$$\lim_{a\to\infty} \frac{\Gamma_m(a)\, a^{pm}}{\Gamma_m(a+p)} = 1$$

we have the required confluence.

7. Distribution of Bartellet's Criterion

In this section, we have applied the theory of ψ function in solving a problem of multivariate statistical analysis.

Let A_1 and A_2 be two $m \times m$ symmetric matrices distributed according to Wishart distributions $W(A_1/\Sigma_1, n_1)$ and $W(A_2/\Sigma_2, n_2)$. Bartellet's criterion for testing the hypothesis; $H : \Sigma_1 = \Sigma_2$, is given by

$$V_1 = \frac{(\det A_1)^{\frac{1}{2}n_1} (\det A_2)^{\frac{1}{2}n_2}}{\det (A_1 + A_2)^{\frac{1}{2}(n_1 \cdot n_2)}}$$

We shall find out the moments of V_1 when the hypothesis H is not true. We note that Anderson[2] has simply stated that the moments of V_1 depend upon the roots $\det(\Sigma_1 - \lambda\Sigma_2) = 0$. But we shall precisely find the moments of V_1 and then it's distribution.

We can express the hth moment V_1 as of

$$\mathscr{E}(V_1)^h = \mathscr{E}\left(\frac{(\det A_1)^{\frac{1}{2}n_1 h} (\det A_2)^{\frac{1}{2}n_2 h}}{\det (A_1 + A_2)^{\frac{1}{2}(n_1 + n_2)h}} \right)$$

$$= \int_{A_1 > 0} \int_{A_2 > 0} W(A_1/\Sigma_1, n_1)\, W(A_2/\Sigma_2, n_2)\, (\det A_1)^{\frac{1}{2}n_1 h} \quad (\det A_2)^{\frac{1}{2}n_2 h}$$

$$\det (A_1 + A_2)^{-\frac{1}{2}(n_1 + n_2)h}\, dA_1\, dA_2$$

$$= \frac{1}{\Gamma_m\left(\frac{n_1}{2}\right) \Gamma_m\left(\frac{n_2}{2}\right) (\det 2\,\Sigma_1)^{\frac{1}{2}n_1} (\det 2\,\Sigma_2)^{\frac{1}{2}n_2}} \int_{A_1 > 0} \int_{A_2 > 0} \operatorname{etr}\left(-\frac{1}{2}\,\Sigma_1^{-1} A_1 \right)$$

$$\times \operatorname{etr}\left(-\frac{1}{2}\,\Sigma_2^{-1} A_2 \right) (\det A_1)^{\frac{1}{2}(n_1 + n_1 h) - p} (\det A_2)^{\frac{1}{2}(n_2 + n_2 h) - p}$$

$$\cdot \det (A_1 + A_2)^{-\frac{1}{2}(n_1 + n_2)h}\, dA_1\, dA_2.$$

$$\left(\text{Since } W(A/\Sigma, n) = \frac{1}{\Gamma_m\left(\frac{n}{2}\right) (\det\ 2\,\Sigma)^{\frac{1}{2}n}} \quad \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1} A \right) (\det A)^{\frac{1}{2}n - p} \right)$$

Now, integrating over $A_2 > 0$, from (5), we have

$$\mathscr{E}(V_1)^h = \frac{\Gamma_m\left(\frac{1}{2}\, n_2 + n_2 h \right)}{\Gamma_m\left(\frac{n_1}{2}\right) \Gamma_m\left(\frac{n_2}{2}\right) (\det 2\,\Sigma_1)^{\frac{1}{2}n_1} (\det 2\Sigma_2)^{\frac{1}{2}n_2}} \int_{A_1 > 0} \operatorname{etr}\left(-\frac{1}{2}\,\Sigma_1^{-1} A^1 \right)$$

$$(\det A_1)^{\frac{1}{2}(n_1 + n_2) - p}$$

$$\cdot\, \psi\left(\frac{1}{2}\,(n_2 + n_2 h);\ \frac{1}{2}(n_2 - n_1 h) - p;\ \frac{1}{2}\,\Sigma_2^{-1} A_1 \right) dA_1. \qquad \ldots(22)$$

In view of the formula (*cf.* Joshi[8] (6.26)) :

$$\int_{\Lambda>0} \operatorname{etr}(-\Lambda R)\operatorname{etr}(-\Lambda)\,\psi(\beta-\alpha,\gamma-\alpha+p;\Lambda)(\det\Lambda)^{\gamma-p}\,d\Lambda$$

$$= \frac{\Gamma_m(\alpha)\Gamma_m(\gamma)}{\Gamma_m(\beta)}\,{}_2F_1(\alpha,\gamma,\beta;-R); \qquad \ldots(23)$$

(where ${}_2F_1(\alpha,\gamma;\beta;-R)$ is hypergeometric function of matrix argument defined by Herz[6]), eqn. (22) reduces to

$$\mathscr{E}(V_1)^h = \frac{\beta_m\left(\frac{n_2+n_2h}{2},\frac{n_1+n_1h}{2}\right)}{\beta_m\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\det\left(\Sigma_1^{-1}\Sigma_2\right)^{\frac{1}{2}n_2h}$$

$$\times\ {}_2F_1\left(\frac{n_2+n_2h}{2},\frac{(n_1+n_2)h}{2};\frac{(h+1)(n_1+n_2)}{2};E-\Sigma_2\Sigma_1^{-1}\right) \qquad \ldots(24)$$

where β_m is generalized beta function of Siegel, and

$$\beta_m(a,b) = \frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)}$$

(*cf.* Herz[6]).

Equation (24) gives hth moment of V_1 when the hypothesis $H;\ \Sigma_1=\Sigma_2$ is not true. If H is true i.e., $\Sigma_1=\Sigma_2$, then (25) reduces to

$$\mathscr{E}(V_1)^h = \frac{\beta_m\left(\frac{1}{2}(n_2+n_2h),\frac{1}{2}(n_1+n_1h)\right)}{\beta_m\left(\frac{1}{2}n_1,\frac{1}{2}n_2\right)}. \qquad \ldots(25)$$

which is same as given by Anderson[2] [10.4 (4)].

For the hypergeometric function of matrix argument, Herz[6] has given following integral representation.

$${}_2F_1(\alpha,\beta,\gamma;Z) = \frac{\Gamma_m(\gamma)}{\Gamma_m(\alpha)\Gamma_m(\beta)}\int_0^E \det(E-RZ)^{-\beta}(\det R)^{\alpha-p}$$

$$\det(E-R)^{\gamma-\alpha-p}\,dR. \qquad \ldots(26)$$

Now in view of (26), (24) reads

$$\mathscr{E}(V_1)^h = \frac{1}{\beta_m\left(\frac{n_1}{2},\frac{n_2}{2}\right)}\cdot\int_0^E [\det((E-R)\Sigma_1+R\Sigma_1)^{\frac{n_1+n_2}{2}}$$

(*equation continued on p.* 636)

$$\times \det((E-R)\,\Sigma_1)^{\frac{n_1}{2}} (\det R\Sigma_2)^{\frac{n_2}{2}}\Big]^h \det(E-R)^{\frac{n_1}{2}-p}$$

$$(\det R)^{\frac{n_2}{2}-p} \, dR \qquad \ldots(27)$$

Thus we see that, when H is not true, V_1 is distributed according to

$$[\det((E-R)\,\Sigma_1 + R\Sigma_2)]^{\frac{n_1+n_2}{2}} \det(E-R)\,\Sigma_1)^{\frac{n_1}{2}} (\det R\Sigma_2)^{\frac{n_2}{2}},$$

where R is distributed in the region; $0 < R < E$ as a generalized beta variate with *p. d. f.*

$$\frac{1}{\beta_m\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \det(E-R)^{\frac{n_1}{2}-p} (\det R)^{\frac{n_2}{2}-p} : 0 < R < E.$$

If H is true V_1 is distributed as $\det(E-R)^{\frac{n_1}{2}} (\det R)^{\frac{n_2}{2}}$ where R is distributed as before.

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