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CHARGED ANALOGUE OF BUCHDAHL'S GENERAL RELATIVISTIC POLYTROPE

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We present here a nonsingular solution of Einstein-Maxwell field equations representing a distribution of charged fluid which in the absence of charge reduces to Buchdahl's solution describing the relativistic analogue of the classical polytrope of index 5. Inside the sphere the matter density and pressure are positive and the charge density real. At large distances from the centre these parameters becomes vanishingly small and field goes over to the Reissner-Nordström field.

INTRODUCTION

Buchdahl[1] obtained a static nonsingular solution of the general relativistic gravitational field equations describing an infinite spherical distribution of perfect fluid which has an equation of state resembling the classical polytropic equation of index 5. The gravitational field asymptotically goes over to Schwarzschild's exterior field at large distances from the centre. In the recent years it has been a problem of general interest to find spherically symmetric solution of Einstein-Maxwell field equations[6] describing physically meaningful distributions of charged perfect fluid. Solutions have been obtained[3,5] which in the case of vanishingly small charge degenerate into Schwarzschild's uniform density solution. Buchdahl[2] has obtained a solution which he calls charged analogue of Schwarzschild's interior solution as the two solutions are conformally flat. Another interesting solution[4] corresponds to an infinite distribution of charged perfect gas with a singular central region. The purpose of the present investigation is to solve Einstein-Maxwell field equations such that the new solution is a natural extension of Buchdahl's solution.

SOLUTION OF EINSTEIN-MAXWELL FIELD EQUATIONS AND ITS PROPERTIES

The interior gravitational field of the model is given by

$$ds^2 = A^2(r)\, c^2\, dt^2 - B^2(r)\,[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \qquad \ldots(1)$$

where

$$A = (1-f^2 + g^2)/(1 + f)^2 - g^2, \; B = (1 + f)^2 - g^2 \qquad \ldots (2a)$$

$$f = \alpha (1 + \beta^2 r^2)^{1/2}, \; g = \gamma (1 + \delta^2 r^2)^{-1/2} \qquad \ldots (2b)$$

α, β, γ and δ are non-negative constants.

The pressure, matter density and charge density distribution are given by

$$16 \pi G p/c^4 = - (rh'' + 3h')/r (1-h) (1 + h + 2f)^2 \qquad \ldots (3)$$

$$16 \pi G \rho/c^2 = - [(4r (h'' + 2f'') + 8 (h' + 2f')) (1 + h + 2f) + rX (1 + h + 2f)^2 - 2r (h' + 2f')^2]/r (1 + h + 2f)^4 \qquad \ldots (4)$$

$$4\pi (G/2)^{1/2} \sigma/c^2 = \eta X^{-1/2} [(4X + rX') (1 + h + 2f) + 2r (h' + 2f') X]/4 (1 + h + 2f)^3 \qquad \ldots (5)$$

where

$$h \equiv f^2 - g^2 \qquad \ldots (6)$$

$$X \equiv [(-rh'' + h') (1 + h + 2f)^2 + 4rh' (h' + 2f') (1 + h + 2f) + 2r (h' + 2f')^2 (1-h)]/r (1-h) (1 + h + 2f)^2 \qquad \ldots (7)$$

and

$$\eta = \pm 1. \qquad \ldots (8)$$

Here and in what follows primes denotes differentiation with respect to r and the subscript o indicates the central value. The non vanishing component of the electromagnetic field tensor are given by

$$F^{01} = - F^{10} = c (X/2G)^{1/2} (1 + h + 2f)^{-1} (1-h)^{-1} \qquad \ldots (9)$$

and the total charge Q (r) within the sphere of radius r is given by

$$2GQ^2/c^4 r^4 = X (1 + h + 2f)^2. \qquad \ldots (10)$$

If we put $\gamma = 0$ the solution reduces to that given earlier by Buchdahl[1]. At large distances from the centre we obtain

$$p \simeq 0, \; \rho \simeq 0, \; dp/d\rho \simeq 0 \qquad \ldots (11)$$

$$Q \simeq 2c^2 \gamma/(G)^{1/2} \delta \text{ and } \sigma \simeq - \eta c^2 \alpha \gamma/\pi (G)^{1/2} \beta \delta r^4.$$

If we identify

$$\alpha/\beta = Gm/2c^2,\ \gamma/\delta = (G)^{1/2} \in /2c^2. \qquad ...(12)$$

We find that the solution passes over asymptotically to the Reissner-Nordström metric as $r \to \infty$. Like Buchdahl's uncharged model, the density and adiabatic sound speed vanish simultaneously with pressure. We also note that at large distances the charge is of opposite kind to that in the centre region.

An interesting feature of the solution may be demonstrated by taking $\delta = \beta$. In this case the pressure, matter density and charge density are given by (3), (4) and (5). At the centre the realistic conditions $p_0 > 0$, $\rho_0 > 0$ are expressed by the inequalities

$$0 < h_0 < 1. \qquad ...(13)$$

From the first inequality it is evident that for the model under consideration $Gm^2 > \epsilon^2$. The second inequality puts a limit on the upper bound of β :

$$\beta < 2c^2/(G\,(Gm^2 - \epsilon^2))^{1/2}. \qquad ...(14)$$

At the centre the square of the adiabatic sound speed is given by

$$(dp/d\rho)_{r=0} = 2c^2\, h_0\,(1 + h_0 + 2\alpha)\,(1 + (1 + 2\alpha)\,(2-h_0))/3\,(1-h_0)^2\,(5\alpha + 6h_0 + 4\,\alpha^2 + 5\alpha h_0) \qquad ...(15)$$

which is always positive, in view of (13). For a causal central distribution the condition is

$$4\,(h_0 + \alpha^2)\,(10\,h_0 - 5\,h_0^2 - 3) + \alpha\,(-15 + 35\,h_0 + 15\,h_0^2 - 19\,h_0^3) \leqslant 0 \qquad ...(16)$$

which is satisfied if $h_0 \leqslant 1 - 2/(10)^{1/2}$. Also for a physically meaningful model the condition $p \leqslant c^2\,\rho$, which imposes an upper bound on the rigidity of the equation of state, is obeyed every where within the sphere. The condition will be satisfied at the centre if $h_0^2 \leqslant (3 - 5h_0)\,\alpha/4$ which is possible if $h_0 < 3/5$. The stronger condition $p_0 \leqslant c^2\,\rho_0/3$ is obeyed if $h_0^2 \leqslant (1 - 3h_0)\,\alpha/2$ which implies that $h_0 < 1/3$. It can be shown that the pressure, matter density and charge density are maximum at the centre and fall monotonically as r increases in an infinitesimal neighbourhood of the centre.

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