

COHOMOLOGY OF LIE ALGEBRA BUNDLES AND ITS APPLICATIONS

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We define the cohomology of a Lie algebra bundle with coefficients in a module bundle. We prove that the cohomology groups vanish when the Lie algebra bundle is semisimple. Using this result we prove the complete reducibility of a module bundle over a semisimple Lie algebra bundle.

§1. A Lie algebra bundle, for short a Lie bundle, as introduced by A. Douady and M. Lazard is a vector bundle $\xi = (\xi, p, \Sigma)$ together with a morphism $\theta : \xi \oplus \xi \rightarrow \xi$ inducing a Lie algebra structure on each fibre³ ξ_x .

A locally trivial Lie bundle ξ is a vector bundle in which each fibre is a Lie algebra and further there exist an open cover $\{U_\alpha\}$ of Σ , Lie algebras $\{L_\alpha\}$ and Lie bundle isomorphisms⁴ $\varphi_\alpha : U_\alpha \times L_\alpha \rightarrow p^{-1}(U_\alpha)$. In general every locally trivial Lie bundle is a Lie bundle but the converse need not be true⁷, whereas a Lie bundle in which all fibres are isomorphic as Lie algebras, is a locally trivial Lie bundle⁵. Also a semisimple Lie bundle, that is a Lie bundle in which each fibre is a semisimple Lie algebra, is locally trivial⁶.

A vector bundle $\eta = (\eta, q, \Sigma)$ is said to be a ξ -module bundle if there exists a morphism $\rho : \xi \oplus \eta \rightarrow \eta$ inducing a ξ_x -module structure on each fibre η_s . By a representation of the Lie bundle (ξ, p, Σ) on a vector bundle (η, q, Σ) we mean a Lie bundle morphism $\rho : \xi \rightarrow \text{End } \eta$ where $\text{End } \eta$ is the Lie bundle whose fibres are the endomorphisms of the vector space⁷ $\eta_s, s \in \Sigma$. A representation induces a module structure and conversely.

If η and η' are two ξ -module bundles, then by a ξ -module bundle morphism $f : \eta \rightarrow \eta'$, we mean a vector bundle morphism in which for each $s \in \Sigma, f_s : \eta_s \rightarrow \eta'_s$ is a ξ_s -module homomorphism.

A vector subbundle (η', q, Σ) of a ξ -module (η, q, Σ) is said to be a submodule bundle of η , if the module structure $\rho : \xi \oplus \eta \rightarrow \eta$ maps $\xi \oplus \eta'$ into η' .

A ξ -module bundle (η, q, Σ) is said to be simple if it has no proper non zero sub-module bundles.

In this paper we give a cochain complex using which the cohomology of a Lie bundle with coefficients in a module bundle can be found. This helps us in proving the complete reducibility of a module bundle over a semisimple Lie bundle.

§2. *Notations and Terminology*—All the vector spaces considered in this paper are real vector spaces, and R denotes the one dimensional vector space of real numbers. The topological space Σ is the base space for all the bundles considered. The symbol ξ denotes the Lie bundle $\xi = (\xi, p, \Sigma, \theta)$ where $\theta : \xi \oplus \xi \rightarrow \xi$ is the Lie structure on ξ .

If $\eta = (\eta, q, \Sigma)$ and $\eta' = (\eta', q', \Sigma)$ are two vector bundles, we denote by $\text{Mor}(\eta, \eta')$, the collection of all vector bundle morphisms from η to η' , and by $\text{Hom}(\eta, \eta')$, the vector bundle $\bigcup_{s \in \Sigma} \text{Hom}(\eta_s, \eta'_s)$ whose fibres are the vector space of linear transformations from η_s to η'_s .

For any vector bundle ξ , $\Gamma(\xi)$ denotes the real vector space of all sections of ξ . We have⁴ that $\text{Mor}(\eta, \eta')$ is isomorphic to $\Gamma(\text{Hom}(\eta, \eta'))$. Given the topological space Σ , let $C(\Sigma)$ denote the algebra of continuous real valued functions defined on Σ . The space $\text{Mor}(\eta, \eta')$ is a $C(\Sigma)$ -module.

Let ξ and η be two vector bundles. $M^p(\xi, \eta)$ denotes the collection of all skew-symmetric maps from $\xi^p = \xi \oplus \xi \oplus \dots \oplus \xi$ (p times) to η , for $p \geq 1$ and let $M^0(\xi, \eta)$ be $\text{Mor}(\Sigma \times R, \eta)$. The vector space $M^p(\xi, \eta)$ is a $C(\Sigma)$ -module⁴.

§3. We introduce the notion of cohomology in this section.

Definition 3.1—Let ξ be a Lie bundle and η a ξ -module bundle with representation $\rho : \xi \rightarrow \text{End } \eta$.

Let us identify $M^0(\xi, \eta)$ with $\Gamma(\eta)$. Given $A \in M^0(\xi, \eta)$, consider the function $f : \xi \rightarrow \eta$ given by $f(x) = \rho(x)(A(s))$ if $x \in \xi_s$. The function f is a morphism being the combination of the following continuous maps :

$$\begin{array}{c} \xi \xrightarrow{(Id, p)} \xi \times \Sigma \xrightarrow{(Id, A)} \xi \oplus \eta \xrightarrow{\rho} \eta \\ x \rightarrow (x, p(x) = s) \rightarrow (x, A(s)) \rightarrow \rho(x)(A(s)). \end{array}$$

Hence $f \in M^1(\xi, \eta)$ and we can define a linear map

$$\delta^0 : M^0(\xi, \eta) \rightarrow M^1(\xi, \eta) \text{ by } \delta^0(A) = f$$

consider $f \in M^n(\xi, \eta)$ and let $g : \xi^{n+1} \rightarrow \eta$ be defined as

$$g(x_0, x_1, \dots, x_n) = \sum_i (-1)^i \rho(x_i) f(x_0, \dots, \hat{x}_i, \dots, x_n) \\ + \sum_{i < j} (-1)^{i+j} f(\theta(x_i, x_j), x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n)$$

where \hat{x}_i means that the i th term is omitted. Obviously $g \in M^{n+1}(\xi, \eta)$ and we define $\delta^n : M^n(\xi, \eta) \rightarrow M^{n+1}(\xi, \eta)$ by $\delta^n(f) = g$.

Then $\{M^n(\xi, \eta), \delta^n\}$ forms a cochain complex and we define the n th cohomology group of ξ with coefficients in η by $H^n(\xi, \eta) = \ker \delta^n / \text{Im } \delta^{n-1}$: We call δ^n , the differentials.

Proposition 3.2—Let $0 \rightarrow \eta_1 \xrightarrow{\alpha} \eta_2 \xrightarrow{\beta} \eta_3 \rightarrow 0$ be a short exact sequence of ξ -module bundles over a compact Hausdorff space Σ . Then there exists a connecting map, $w_n : H^n(\xi, \eta_3) \rightarrow H^{n+1}(\xi, \eta_1)$ such that

$$0 \rightarrow H^0(\xi, \eta_1) \xrightarrow{\alpha_0} H^0(\xi, \eta_2) \xrightarrow{\beta_0} H^0(\xi, \eta_3) \\ \xrightarrow{w_0} H^1(\xi, \eta_1) \rightarrow \dots \rightarrow H^n(\xi, \eta_1) \xrightarrow{\alpha_n} \\ H^n(\xi, \eta_2) \xrightarrow{\beta_n} H^n(\xi, \eta_3) \xrightarrow{w_n} H^{n+1}(\xi, \eta_1) \rightarrow \dots$$

is exact where α_n and β_n are given by

$$\alpha_n(f + \text{Im } \delta_1^{n-1}) = \alpha f + \text{Im } \delta_2^{n-1}$$

and

$$\beta_n(f + \text{Im } \delta_2^{n-1}) = \beta f + \text{Im } \delta_3^{n-1}.$$

PROOF: The sequence $0 \rightarrow \eta_1 \xrightarrow{\alpha} \eta_2 \xrightarrow{\beta} \eta_3 \rightarrow 0$ being an exact sequence of ξ -module bundles, is an exact sequence of vector bundles over the compact Hausdorff space Σ . Hence there exists an isomorphism $\eta_2 = \eta_1 \oplus \eta_3$ [Atiyah¹, Cor. 1.4.11, p. 25]. This gives a splitting vector bundle morphism $\gamma : \eta_3 \rightarrow \eta_2$.

Now we can define $w_n : H^n(\xi, \eta_3) \rightarrow H^{n+1}(\xi, \eta_1)$ by

$w_n(f + \text{Im } \delta_3^{n-1}) = \alpha^{-1} \delta_2^n(\gamma f) + \text{Im } \delta_1^n$. Then w_n is well defined and the sequence,

$$0 \rightarrow H^0(\xi, \eta_1) \rightarrow H^0(\xi, \eta_2) \rightarrow H^0(\xi, \eta_3) \rightarrow H^1(\xi, \eta_1) \rightarrow \dots$$

is exact.

Remark 3.3 : The cochain complex in 1.1 can be viewed in a different way. Let $\text{Hom}(\xi^p, \eta) = \bigcup_{s \in \Sigma} \text{Hom}(\xi_s^p, \eta_s)$ be the vector bundle, where $\text{Hom}(\xi_s^p, \eta_s)$ is the vector space of all p skew-symmetric maps from ξ_s^p to η_s .

We can define the vector bundle morphisms $\Delta^0 : \eta \rightarrow \text{Hom}(\xi, \eta)$ by $\Delta^0(a) = \rho_s(-, a)$ for $a \in \eta_s$. We define

$$\Delta^n : \text{Hom}(\xi^n, \eta) \rightarrow \text{Hom}(\xi^{n+1}, \eta)$$

by

$$\begin{aligned} \Delta^n(f)(x_0, \dots, x_n) &= \sum_i (-1)^i \rho_s(x_i) f(x_0, \dots, \hat{x}_i, \dots, x_n) \\ &+ \sum_{i < j}^i (-1)^{i+j} f(\theta(x_i, x_j), x_0, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n) \end{aligned}$$

if $f \in \text{Hom}(\xi_s^n, \eta_s)$.

These morphisms Δ^n induce the mappings $\delta^n : \Gamma(\text{Hom}(\xi^n, \eta)) \rightarrow \Gamma(\text{Hom}(\xi^{n+1}, \eta))$. Our cochain complex defined in 1.1 is nothing but the complex $\{\Gamma(\text{Hom}(\xi^n, \eta)), \delta^n\}$. So the cohomology defined in Definition 3.1 is the space of sections of the fibre bundle whose fibres are the cohomology of the fibres. That is $H^n(\xi, \eta) = \Gamma(\bigcup_{s \in \Sigma} H^n(\xi_s, \eta_s))$. Hence the following :

Proposition 3.4—The cohomology groups $H^n(\xi, \eta), n \geq 1$, vanish, when ξ is semisimple.

§4. Now we give an application of the above proposition.

We shall first observe the following :

Remark 4.1 : If η_1 and η_2 are ξ -module bundles with representations ρ_1 and ρ_2 , then $\text{Hom}(\eta_1, \eta_2)$ can be made into a ξ -module by defining $\rho^* : \xi \oplus \text{Hom}(\eta_1, \eta_2) \rightarrow \text{Hom}(\eta_1, \eta_2)$ by $\rho^*(x, f) = g$ for $x \in \xi_s, f \in \text{Hom}((\eta_1)_s, (\eta_2)_s)$ where $g \in \text{Hom}((\eta_1)_s, (\eta_2)_s)$ is given by $g(a) = \rho_2(x)(f(a)) - f(\rho_1(x)(a))$.

Remark 4.2 : To show that a ξ -module bundle η is completely reducible, it is enough to prove that, given any proper submodule bundle (η', q, Σ) of η , the exact sequence $0 \rightarrow \eta' \rightarrow \eta \rightarrow \eta/\eta' \rightarrow 0$ splits as a sequence of ξ -modules. That is there exists a ξ -module bundle morphism $f : \eta \rightarrow \eta'$ such that $f|_{\eta'} = \text{identity}$.

Theorem 4.3 : If ξ is a semisimple Lie bundle, then every module bundle over ξ is completely reducible, when the base space Σ is compact Hausdorff.

PROOF : Let η' be a submodule bundle of η . Then we obtain the exact sequence

$$0 \rightarrow \eta' \xrightarrow{i} \eta \xrightarrow{\pi} \eta/\eta' \rightarrow 0$$

of ξ -module bundles, where i is the inclusion and π the canonical projection.

This gives rise to the exact sequence

$$0 \rightarrow \text{Hom}(\eta/\eta', \eta') \xrightarrow{\pi_*} \text{Hom}(\eta, \eta') \xrightarrow{i_*} \text{Hom}(\eta', \eta') \rightarrow 0$$

of vector bundles over Σ . By Remark 4.1, this can be considered as an exact sequence of ξ -module bundles over Σ . The base space Σ being compact Hausdorff, by Proposition 3.2, we obtain a long exact cohomology sequence

$$0 \rightarrow H^0(\xi, \text{Hom}(\eta/\eta', \eta')) \xrightarrow{(\pi_*)_0} H^0(\xi, \text{Hom}(\eta, \eta')) \xrightarrow{(i_*)_0} H^0(\xi, \text{Hom}(\eta', \eta')) \xrightarrow{w_0} H^1(\xi, \text{Hom}(\eta', \eta')) \rightarrow \dots$$

Since ξ is semisimple $H^1(\xi, \text{Hom}(\eta/\eta', \eta')) = 0$. Hence $(i_*)_0$ is onto.

Consider the section $I : \Sigma \rightarrow \text{Hom}(\eta', \eta')$ given by $I(s) = \text{identity on } \eta'_s$ for each $s \in \Sigma$. Then $I \in H^0(\xi, \text{Hom}(\eta', \eta'))$ and there exists $A \in H^0(\xi, \text{Hom}(\eta, \eta'))$ such that $(i_*)_0(A) = I$. We can define $\mu : \eta \rightarrow \eta'$ by $\mu(a) = (A(s))(a)$ if $a \in \eta_s$. Then μ gives the required splitting ξ -module bundle morphism. Hence the theorem.

Remark 4.4 : By following the method as in Bourbaki² (Theorem 1, p. 357) the above theorem can be proved for semisimple Lie group bundles, where by a semisimple Lie group bundle we mean a Lie group bundle³ in which each fibre is a semisimple Lie group.

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