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# SOME PROPERTIES OF A SEMI-SYMMETRIC METRIC CONNECTION IN A RIEMANNIAN MANIFOLD

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The present paper deals with a Riemannian Manifold $M$ admitting a Semi-Symmetric metric connection $\bar{\nabla}$ whose torsion $T(X, Y)$ is given by

$$T(X, Y) = \pi(Y) X - \pi(X) Y$$

where $\pi$ is a differential 1-form on $M$ and $X, Y \in \mathfrak{X}(M)$, the set of all differentiable vectorfields on $M$. Some properties of the manifold which depends upon the Ricci tensor of the semi-symmetric metric connection under certain conditions, are studied.

§1. Let $M$ be an $n$-dimensional Riemannian manifold with the Riemann metric $g$, and let $\nabla$ denote the Levi-Civita connection in $M$.

Let $\bar{\nabla}$ be a linear connection in $M$ whose torsion tensor[2]

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

is given by

$$T(X, Y) = \pi(Y) X - \pi(X) Y \qquad .. (1.1)$$

where $\pi$ is a differentiable 1-form on $M$ and $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the set of all differentiable vectorfields on $M$. Then $\bar{\nabla}$ is a semi-symmetric connection[1-3] in $M$. $\bar{\nabla}$ is called a metric connection[2] if it satisfies

$$\bar{\nabla} g = 0. \qquad .. (1.2)$$

In the present paper we shall consider $M$ as an $n$-dimensional Riemannian manifold admitting a semi-symmetric metric connection satisfying (1.1) and (1.2). In section 2, a necessary and sufficient condition has been deduced for the Ricci tensor of the semi-symmetric metric connection $\bar{\nabla}$ to be symmetric, when $n \geqslant 3$.

In section 3, a necessary and sufficient condition has been deduced for the Ricci tensor of $\bar{\nabla}$ to be skew-symmetric under certain condition. It has been shown that if the Ricci tensor fo $\bar{\nabla}$ is skew-symmetric, then Weyl's conformal curvature tensors for $\bar{\nabla}$ and $\nabla$ are equal.

§2. If $\bar{\nabla}$ is a semi-symmetric metric connection in $M$, then there exists a vector-field $P$ in $\mathfrak{X}(M)$ such that[4]

$$\bar{\nabla}_X Y = \nabla_X Y + \pi(Y) X - g(X, Y) P \qquad \ldots(2.1)$$

where

$$g(X, P) = \pi(X). \qquad \ldots(2.2)$$

If $\bar{K}(X, Y) Z$ is the curvature of $\bar{\nabla}$ and if $K(X, Y) Z$ is the Riemann curvature tensor in $M$, then[4]

$$\bar{K}(X, Y) Z = K(X, Y) Z - \alpha(Y, Z) X + \alpha(X, Z) Y - g(Y, Z) AX + g(X, Z) AY \qquad \ldots(2.3)$$

where $\alpha$ is a tensorfield of type (0, 2) given by

$$\alpha(X, Y) = (\nabla_X \pi)(Y) - \pi(X) \pi(Y) + \frac{1}{2} \pi(P) g(X, Y) \qquad \ldots(2.4)$$

and $A$ is a tensorfield of type (1, 1) given by

$$g(AX, Y) = \alpha(X, Y). \qquad \ldots(2.5)$$

If $\bar{S}(X, Y)$ and $S(X, Y)$ denote the Ricci tensors of the connections $\bar{\nabla}$ and $\nabla$ respectively, then from (2.3) we get

$$\bar{S}(X, Y) = S(X, Y) - (n-2) g(Y, \nabla_X P) + (n-2) \pi(X) \pi(Y) - (n-2) \pi(P) g(X, Y) - g(X, Y) \sum_{i=1}^{n} (g(\nabla_{X_i} P, X_i)) \qquad \ldots(2.6)$$

where $X_1, \ldots\ldots, X_n$ is an orthonormal basis of $\mathfrak{X}(M)$. From (2.6) we get, by virtue of the symmetry of $S(X, Y)$

$$\bar{S}(X, Y) - \bar{S}(Y, X) = (n-2)[(\nabla_Y \pi)(X) - (\nabla_X \pi)(Y)]. \qquad \ldots(2.7)$$

Hence if $\bar{S}(X, Y)$ is symmetric, then, for $n \geqslant 3$,

$$(\nabla_Y \pi)(X) - (\nabla_X \pi)(Y) = 0 \qquad \ldots(2.8)$$

which shows that $d\pi = 0$, that is, $\pi$ is closed.

Conversely, if $\pi$ is a closed $l$-form, then (2.8) holds, and hence

$$\bar{S}(X, Y) = \bar{S}(Y, X).$$

We state the following :

*Theorem* 2.1.—In order that a Riemannian manifold of dimension $n$ ($n \geqslant 3$) admits a semi-symmetric metric connection whose Ricci tensor is symmetric, it is necessary and sufficient that the $l$-form $\pi$ is closed.

§3. In this section we shall consider

$$(\nabla_Y \pi)(X) + (\nabla_X \pi)(Y) = 0. \quad \ldots (3.1)$$

Then $P$ is a Killing vector field[5] satisfying

$$g(\nabla_X P, Y) + g(\nabla_Y P, X) = 0. \quad \ldots (3.2)$$

From (2.6) we get, by virtue of (3.2)

$$\bar{S}(X, Y) + \bar{S}(Y, X) = 2S(X, Y) + 2(n-2)[\pi(X)\pi(Y) - \pi(P) g(X, Y)]. \quad \ldots (3.3)$$

If $\bar{S}(X, Y)$ is skew-symmetric the left hand side of (3.3) vanishes, and we get

$$S(X, Y) = (n-2)[\pi(P) g(X, Y) - \pi(X)\pi(Y)]. \quad \ldots (3.4)$$

On the other hand if $S(X, Y)$ is given by (3.4), then from (3.3) we get

$$\bar{S}(X, Y) + \bar{S}(Y, X) = 0.$$

*Theorem* 3.1—If a Riemannian manifold of dimension $n$ ($n \geqslant 3$) admits a semi-symmetric metric connection $\bar{\nabla}$ with $P$ as a Killing vectorfield, then a necessary and sufficient condition for the Ricci tensor of $\bar{\nabla}$ to be skew-symmetric is that the Ricci tensor of the Levi-Civita connection $\nabla$ is given by (3.4).

Also, if $S(X, Y) = 0$, from (3.4) we get $\pi(P) = 0$ for $n \geqslant 3$. Which is not possible.

We can state :

*Corollary* 3.1—If a Riemannian manifold of dimension $n$ ($n \geqslant 3$) admits a semi-symmetric metric connection whose Ricci-tensor is skew-symmetric and if $P$ is a Killing vectorfield then the manifold cannot be Ricci-flat.

Again, for $n = 2$, from (3.3) we see that if $\bar{S}(X, Y)$ is skew-symmetric then $S(X, Y) = 0$ and conversely.

*Theorem* 3.2—If a Riemannian manifold of dimension 2 admits a semi-symmetric metric connection $\bar{\nabla}$ and if $P$ is a Killing vectorfield, then a necessary and sufficient condition for the Ricci-tensor of $\bar{\nabla}$ is skew-symmetric is that the manifold is Ricci-flat.

If $r$ denote the scalar curvature of the Levi-Civita connection then from (3.4)

$$\pi(P) = \frac{r}{(n-1)(n-2)}, \quad n \geqslant 3. \qquad \ldots(3.5)$$

Applying (3.5) to (3.4) we get

$$\pi(X)\,\pi(Y) = \lambda(X, Y) + \frac{1}{2}\,\pi(P)\,g(X, Y) \qquad \ldots(3.6)$$

where

$$\lambda(X, Y) = -\frac{S(X, Y)}{n-2} + \frac{r}{2(n-1)(n-2)}\,g(X, Y). \qquad \ldots(3.7)$$

Since $\bar{S}(X, Y)$ is skew-symmetric, the scalar curvature for $\bar{\nabla}$ is zero and we get

$$\bar{\lambda}(X, Y) = -\frac{\bar{S}(X, Y)}{n-2} = g(\nabla_X P, Y). \qquad \ldots(3.8)$$

Using (3.6) to (3.8) in (2.4) and applying the result in (2.3) we get

$$\begin{aligned}\bar{K}(X, Y)\,Z = {} & K(X, Y)\,Z - \bar{\lambda}(Y, Z)\,X + \bar{\lambda}(X, Z)\,Y + \lambda(Y, Z)\,X \\ & - \lambda(X, Z)\,Y - \frac{1}{2}\,\pi(P)\,[g(Y, Z)\,X - g(X, Z)\,Y] \\ & - g(Y, Z)\,\nabla_X P + g(X, Z)\,\nabla_Y P + [g(Y, Z)\,\pi(X) \\ & - g(X, Z)\,\pi(Y)]\,P. \qquad \ldots(3.9)\end{aligned}$$

Let us now define two (1,1) tensorfields $\bar{L}$ and $L$ by

$$\bar{L}\,X = \nabla_X P \text{ and } LX = \pi(X)\,P - \frac{1}{2}\,\pi(P)\,X \qquad \ldots(3.10)$$

$$g(\bar{L}\,X, Y) = g(\nabla_X P, Y) = \bar{\lambda}(X, Y) \qquad \ldots(3.11)$$

$$g(L\,X, Y) = \pi(X)\,\pi(Y) - \frac{1}{2}\,\pi(P)\,g(X, Y) = \lambda(X, Y).$$

By virtue of (3.11), we get from (3.9)

$$\bar{C}(X, Y)\,Z = C(X, Y)\,Z \qquad \ldots(3.12)$$

where

$$C(X, Y)Z = K(X, Y)Z - g(LX, Z)Y + g(LY, Z)X + g(Y, Z)LX - g(X, Z)LY \quad .. (3.13)$$

is Weyl's conformal curvature tensor for the Levi-Civita connection and $\bar{C}(X, Y)Z$ is the same for the connection $\bar{\nabla}$. Hence

*Theorem* 3.3—If a Riemannian manifold of dimension $n$ ($n \geqslant 3$) admits a semi-symmetric metric connection $\bar{\nabla}$ whose Ricci tensor is skew-symmetric and if $P$ is a Killing vectorfield, then Weyl's conformal curvature tensor of $\bar{\nabla}$ is equal to that of the Levi-Civita connection.

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