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A NOTE ON A QUEUEING MODEL WITH ARRIVAL RATE DEPENDING UPON THE NATURE OF SERVICE

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This note is concerned with the time-dependent analysis of a limited capacity queueing model with arrival rate depending upon the nature of service available in the system. The service consists of two stages, one is essential (the first stage) while the other may be inessential. The decision to offer the inessential service depends upon the size and recent history of the system. However, if this inessential service is temporarily suspended, the arrival rate of customers diminishes. Laplace transforms (in time) of the different probability generating functions describing the system size under various natures of service, and the expected system size are derived. From thhse the steady state results follow immediately.

1. INTRODUCTION

Scott and Hsia[2] made the steady-state analysis of a queueing model of limited capacity where the arrival pattern is Poisson but not constant throughout, rather depends upon the nature of the service available. They considered the service to be consisting either of a single stage Poisson type which may be termed as essential, or of two stage Poisson type where the second stage is of inessential nature. When the queue length increases to some critical number R, the second stage of service is suspended and it is started again only when the queue length falls to a certain number r ($r < R$). The arrival rate in the system obviously depends on the natures of service available. The arrival rate when both the services are available is greater than the arrival rate when only the essential service is available. This type of situation may arise in the sale season in departmental stores where small presents are given to customers ("inessential service") after the actual purchases ("essential service") to attract more customers. As the sale season is a short one trends may not last long enough for the steady state to reach, hence the need for a time dependent solution.

In the study of time dependent analysis of queueing models with Poisson type arrival and service, usually a set of differential equations for suitable probability generating functions are derived. This set is then reduced to a set of linear equations in the

Laplace transforms of the generating functions which then can be solved. By inversion, these generating functions then can be obtained from which the different probabilities can be derived in principle as a function of time. However, the exact derivation of the time dependent quantities is rather complicated and as such, the steady-state solutions are mostly given. Murari[1] while studying a queueing model with additional service channels (a. s. c's) used this technique to derive the Laplace transforms of different probability generating functions from which the corresponding steady state results were obtained. The a. s. c's considered in Murari[1] are dependent on system size and are pressed into service in stages whenever the queue length attains some critical values. Yoneyama *et al.*[3] modified the model of Murari[1] by considering the a. s. c's as random in the following sense. When the queue length grows to some critical number, a search for another channel is started and the availability time of this channel is assumed to be random. The search continues till the queue length reduces to a certain value less than the previous critical number. If the search is fulfilled, another service channel starts functioning instantaneously.

In the present note, the queueing model of Scott and Hsia[2] is extended to the non-stationary case. Defining generating functions of the four different types of probabilities appearing in the Kolmogorov's forward equation, and then taking Laplace transforms in time, expressions for the Laplace transforms of these generating functions are obtained. The Laplace transform of the expected system size can be derived from these. Derivation of the time-dependent expression for the expected system size is straight forward once the Laplace inversion can be carried out. However this is not attempted here, and instead, we make the parameter of the Laplace transform tend to zero whereby the steady-state part of the expected system size is recovered which agrees completely with that given in Scott and Hsia[2].

2. Statement and Formulation of the Problem

In the queueing model of limited capacity considered here the total service received by a customer comprises of two stages, the first is essential while the second is inessential. The pattern of arrival and both the services is Poissonian. When the number of customers in the system reaches R ($< N$, the capacity of the system), the second stage of service is suspended resulting thereby a drop in the arrival rate of customers from λ_1 to λ_2 ($< \lambda_1$). The second stage of service remains suspended until the system size decreases to r ($0 < r < R$) when both the stages become available and the arrival rate changes back to λ_1. The queue discipline is FCFS, the service periods for the first and second stages are independent random variables with means μ_1^{-1} and μ_2^{-1} respectively. The arrival and service processes are independent.

Let $p\ (t, 0)$ denote the probability that there is no customer in the system at time t, $p\ (t, n; i)$ the probability that there are n ($n = 1, 2, \ldots R-1$) customers in the system

at time t when the customer being served is in the ith stage of service ($i = 1, 2$), second stage of service being available, and $q(t, n; i)$ ($n = r + 1, r + 2, \ldots, N$ for $i = 1$ and $n = R, R + 1, \ldots, N$ for $i = 2$) is the same when the customer being served is in the ith stage of service ($i = 1, 2$), second stage of service being unavailable. It may be noted here that unavailability of the second stage of service does not effect a customer whose second stage of service has already begun. It can be shown that the probabilities satisfy the following set of Kolmogorov's forward equations

$$\frac{\partial p(t, 0)}{\partial t} = -\lambda_1 p(t, 0) + \mu_2 p(t, 1; 2) \qquad \ldots(2.1)$$

$$\frac{\partial p(t, 1; 1)}{\partial t} = -(\lambda_1 + \mu_1) p(t, 1; 1) + \lambda_1 p(t, 0) + \mu_2 p(t, 2; 2) \qquad \ldots(2.2)$$

$$\frac{\partial p(t, n; 1)}{\partial t} = -(\lambda_1 + \mu_1) p(t, n; 1) + \lambda_1 p(t, n-1; 1) + \mu_2 p(t, n + 1; 2)$$

$$(n = 2, 3, \ldots r-1, r + 1, \ldots R-2) \qquad \ldots(2.3)$$

$$\frac{\partial p(t, r; 1)}{\partial t} = -(\lambda_1 + \mu_1) p(t, r; 1) + \lambda_1 p(t, r-1; 1) + \mu_2 p(t, r + 1; 2) + \mu_1 q(t, r + 1; 1) \qquad \ldots(2.4)$$

$$\frac{\partial p(t, R-1; 1)}{\partial t} = -(\lambda_1 + \mu_1) p(t, R-1; 1) + \lambda_1 p(t, R-2; 1); \qquad \ldots(2.5)$$

$$\frac{\partial p(t, 1; 2)}{\partial t} = -(\lambda_1 + \mu_2) p(t, 1; 2) + p(t, 1; 1), \qquad \ldots(2.6)$$

$$\frac{\partial p(t, n; 2)}{\partial t} = -(\lambda_1 + \mu_2) p(t, n; 2) + \lambda_1 p(t, n-1; 2) + \mu_1 p(t, n; 1)$$

$$(n = 2, 3, \ldots, R-1); \qquad \ldots(2.7)$$

$$\frac{\partial q(t, r + 1; 1)}{\partial t} = -(\lambda_2 + \mu_1) q(t, r + 1; 1) + \mu_1 q(t, r + 2; 1), \qquad \ldots(2.8)$$

$$\frac{\partial q(t, n; 1)}{\partial t} = -(\lambda_2 + \mu_1) q(t, n; 1) + \lambda_2 q(t, n-1; 1) + \mu_1 q(t, n + 1; 1),$$

$$(n = r + 2, r + 3, \ldots R-2) \qquad \ldots(2.9)$$

$$\frac{\partial q(t, R-1; 1)}{\partial t} = -(\lambda_2 + \mu_1) q(t, R-1; 1) + \mu_1 q(t, R; 1) + \lambda_2 q(t, R-2; 1) + \mu_2 q(t, R; 2) \qquad \ldots(2.10)$$

$$\frac{\partial q(t, R; 1)}{\partial t} = -(\lambda_2 + \mu_1) q(t, R; 1) + \lambda_1 p(t, R-1; 1) + \lambda_2 q(t, R-1; 1) + \mu_1 q(t, R+1; 1) + \mu_2 q(t, R+1; 2), \quad \ldots (2.11)$$

$$\frac{\partial q(t, n; 1)}{\partial t} = -(\lambda_2 + \mu_1) q(t, n; 1) + \mu_1 q(t, n+1; 1) + \mu_2 q(t, n+1; 2) + \lambda_2 q(t, n-1; 1), (n = R+1, R+2, \ldots N-1) \quad \ldots (2.12)$$

$$\frac{\partial q(t, N; 1)}{\partial t} = -\mu_1 q(t, N; 1) + \lambda_2 q(t, N-1; 1); \quad \ldots (2.13)$$

$$\frac{\partial q(t, R; 2)}{\partial t} = -(\lambda_2 + \mu_2) q(t, R; 2) + \lambda_1 q(t, R-1; 2) \quad \ldots (2.14)$$

$$\frac{\partial q(t, n; 2)}{\partial t} = -(\lambda_2 + \mu_2) q(t, n; 2) + \lambda_2 q(t, n-1; 2), \quad (n = R+1, R+2, \ldots N-1) \quad \ldots (2.15)$$

$$\frac{\partial q(t, N; 2)}{\partial t} = -\mu_1 q(t, N; 2) + \lambda_2 q(t, N-1; 2). \quad \ldots (2.16)$$

Assuming the system to be empty at $t = 0$, we can write the initial conditions as

$$p(0, 0) = 1, \; p(0, n; i) = 0, \; q(0, n; i) = 0, \; n = 0, \; i = 1, 2. \quad \ldots (2.17)$$

3. Time Dependent Analysis

Let us introduce the following probability generating functions (p. g. f)

$$P(t, z; i) = \sum_{n=1}^{R-1} p(t, n; i) z^n \; (i = 1, 2) \quad \ldots (3.1)$$

$$Q(t, z; 1) = \sum_{n=r+1}^{N-1} q(t, n; 1) z^n \quad \ldots (3.2)$$

$$G(t, z; 2) = \sum_{n=R}^{N-1} q(t, n; 2) z^n. \quad \ldots (3.3)$$

Let $\bar{F}(s)$ denote the Laplace transform (L. T.) of $F(t)$ defined by

$$\bar{F}(s) = \int_0^\infty F(t)\, e^{-st}\, dt. \qquad \ldots (3.4)$$

Multiplying (2.2) to (2.15) by appropriate powers of z, summing over n within appropriate limits, using (3.1), to (3.3) and taking L. T.'s we obtain

$$\bar{P}(s, z; i) = \Delta_i(s, z)/\Delta(s, z), \ (i = 1, 2),$$

$$\bar{Q}(s, z; 1) = \delta_1(s, z)/\delta_2(s, z) \qquad \ldots (3.5)$$

$$\bar{Q}(s, z; 2) = \delta_3(s, z)/\delta_4(s, z)$$

where

$$\Delta_1(s, z) = (s + \lambda_1 + \mu_2 - \lambda_1 z)\,[z - \{\lambda_1(1-z) + s\}\, z\, \bar{p}(s, 0) - \lambda_1 z^{R+1}\, \bar{p}(s, R-1; 1) + \mu_1 z^{r+1}\, \bar{q}(s, r+1; 1)] - \lambda_1 \mu_2 z^R\, \bar{p}(s, R-1; 2) \qquad \ldots (3.6)$$

$$\Delta_2(s, z) = \mu_1\,[z - \{\lambda_1(1-z) + s\}\, z\, \bar{p}(s, 0) - \lambda_1 z^{R+1}\, \bar{p}(s, R-1; 1) + \mu_1 z^{r+1}\, \bar{q}(s, r+1; 1)] - \{s + \lambda_1(1-z) + \mu_1\}_1 z^{R+1}\, \bar{p}(s, R-1; 2) \qquad \ldots (3.7)$$

$$\Delta(s, z) = \lambda_1 z^3 - \lambda_1(2\lambda_1 + \mu_1 + \mu_2 + 2s)\, z^2 + \{\lambda_1(\lambda_1 + \mu_1 + \mu_2 + 2s) + s(\mu_1 + \mu_2) + \mu_1 \mu_2\}\, z - \mu_1 \mu_2 \qquad \ldots (3.8)$$

$$\begin{aligned}\delta_1(s, z) = {} & \lambda_1 z^R\, \bar{p}(s, R-1; 1) - \mu_1 z^r\, \bar{q}(s, r+1; 1) - \lambda_2 z^N \\ & \times [\{1 - \{(\lambda_2 + s)/\mu_1\}^{R-r}\}\, \{1 - (\lambda_2 + s)/\mu_1\}^{-1}\, \{(\lambda_2 + s)/\mu_1\}^{N-R-1} \\ & \times \bar{q}(s, r+1; 1) + (\lambda_1/\mu_1)\, \{\lambda_2/(\lambda_2 + \mu_2 + s) \\ & - (\lambda_2 + s)/\mu_1\}^{-1}\, \{\{\lambda_2/(\lambda_2 + \mu_2 + s)\}^{N-R+1} \\ & - \{(\lambda_2 + s)/\mu_1\}^{N-R-1}\, \{\lambda_2/(\lambda_2 + \mu_2 + s) - (\lambda_2 + s)/\mu_1 \\ & + \lambda_2(\lambda_2 + s)/\mu_1\, (\lambda_2 + \mu_2 + s)^{-1}\}]\, \bar{p}(s, R-1; 2)] \\ & + \mu_1 z^{N-1}\, [\{1 - \{(\lambda_2 + s)/\mu_1\}^{R-r}\}\, \{1 - (\lambda_2 + s)/\mu_1\}^{-1}\end{aligned}$$

(equation continued on p. 848)

$$\times \{(\lambda_2 + s)/\mu_1\}^{N-R} \bar{q}(s, r+1; 1) + \lambda_1 (\lambda_2 + s) \mu_1^{-2}$$

$$\times \{\lambda_2/(\lambda_2 + \mu_2 + s) - (\lambda_2 + s)/\mu_1\}^{-1} \{\{\lambda_2/(\lambda_2 + \mu_2 + s)\}^{N-R+1}$$

$$- \{(\lambda_2 + s)/\mu_1\}^{N-R-1} \{\lambda_2/(\lambda_2 + \mu_2 + s) - (\lambda_2 + s)/\mu_1$$

$$+ \lambda_2 (\lambda_2 + s) \{\mu_1 (\lambda_2 + \mu_2 + s)\}^{-1}\} \bar{p}(s, R-1; 2)\}] \quad \ldots(3.9)$$

$$\delta_2 (s, z) = s + \lambda_2 + \mu_1 - \lambda_2 z - \mu_1/z \quad \ldots(3.10)$$

$$\delta_3 (s, z) = \lambda_1 z^R \bar{p}(s, R-1; 2) - \lambda_1 \lambda_2^{N-R} (\lambda_2 + \mu_2 + s)^{-N+R} z^N \bar{p}(s, R-1; 2) \quad \ldots(3.11)$$

$$\delta_4 (s, z) = s + \lambda_2 + \mu_2 - \lambda_2 z. \quad \ldots(3.12)$$

Note that Δ_i $(i = 1, 2)$, δ_j $(j = 1, 2)$ contain only four unknowns, namely $\bar{p}(s, 0)$, $\bar{p}(s, R-1; i)$, $(i = 1, 2)$ and $\bar{q}(s, r+1; 1)$. Since $\bar{P}(s, z; i)$, $\bar{Q}(s, z; i)$ $(i = 1, 2)$ are polynomials regarded as a functions of z only, the numerators in (3.5) must vanish at the zeros of the denominators. Although it appears that nine equations will be obtained by using these conditions but in fact only three of these equations are independent. So the three equations written below in (3.13) together with the equation obtained from the condition that the sum of all probabilities is unity i.e., the sum of the L. T.'s of all probabilities is s^{-1}, are sufficient to determine these four unknowns uniquely.

$$\{s + \lambda_1 (1-z_i) + \mu_2\} [z_i - \{\lambda_1 (1-z_i) + s\} z_i \bar{p}(s, 0) - \lambda_1 z_i^{R+1} \bar{p}(s, R-1; 1) + \mu_1 z_i^{r+1} \bar{q}(s, r+1; 1)] - \lambda_1 \mu_2 z_i^R$$

$$\bar{p}(s, R-1; 2) = 0 \; (i = 1, 2, 3) \quad \ldots(3.13)$$

where z_i (s) are the zeros of $\Delta(z, s)$ $(i = 1, 2, 3)$. Now by inversion, the generating functions defined in (3.1) to (3.3) can be found and hence the different probabilities are determined in principle as functions of time t. However, the actual calculation involved in these steps are rather complicated and are not attempted here. Instead we deduce the results of the steady state case.

4. Steady State Solution

The steady state solution can be derived by using the well known property of the L. T.'s namely,

$$\lim_{t \to \infty} F(t) = \lim_{s \to 0} s \bar{F}(s) \quad \ldots(4.1)$$

if the limit on the left hand side exists. Thus if

$$\lim_{t\to\infty} p\,(t, n; i) = p\,(n; i)$$

and

$$P\,(z; i) = \sum_{n=1}^{R-1} p\,(n; i)\, z^n \;(i = 1, 2), \text{ etc.} \qquad \ldots(4.2)$$

we have

$$\lim_{s\to 0} s\,\bar{p}\,(s, n; i) = p\,(n; i) \text{ and } \lim_{s\to 0} s\,\bar{P}\,(s, z; i) = P\,(z; i), \text{ etc.} \qquad \ldots(4.3)$$

Applying the property (4.1) to equations in (3.5), we have

$$\begin{aligned} &P\,(z; i) = \Delta_i\,(z)/\Delta\,(z),\ (i = 1, 2),\\ &Q\,(z, 1) = \delta_1\,(z)/\delta_2\,(z) \qquad \ldots(4.4)\\ &Q\,(z, 2) = \delta_3\,(z)/\delta_4\,(z) \end{aligned}$$

where

$$\Delta_1\,(z) = \{\lambda_1\,(1-z) - \mu_2\}\,\{\lambda_1 z\,(1-z)\,p\,(0) + \lambda_1\, z^{R+1}\,p\,(R-1; 1) - \mu_1\, z^{r+1}\,q\,(r+1; 1)\} - \lambda_1\,\mu_2\, z^R\,p\,(R-1; 2), \qquad \ldots(4.5)$$

$$\Delta_2\,(z) = \lambda_1\,\{\lambda_1\,(z-1) - \mu_1\}\, z^{R+1}\,p\,(R-1; 2) - \mu_1\,\{\lambda_1\, z\,(1-z)\,p\,(0) + \lambda_1\, z^{R+1}\,p\,(R-1; 1) - \mu_1\, z^{r+1}\,q\,(r+1; 1)\}, \qquad \ldots(4.6)$$

$$\Delta\,(z) = \lambda_1^2\, z^3 - \lambda_1\,(2\lambda_1 + \mu_1 + \mu_2)\, z^2 + \{\lambda_1\,(\lambda_1 + \mu_1 + \mu_2) + \mu_1\,\mu_2\}\, z - \mu_1\,\mu_2, \qquad \ldots(4.7)$$

$$\begin{aligned} \delta_1\,(z) = {} & \lambda_1\, z^R\,p\,(R-1; 1) - \mu_1\, z^r\,q\,(r+1; 1) - \lambda_2\, z^N\,[\{1 - (\lambda_2/\mu_1)^{R-r}\} \\ & \times (1-\lambda_2/\mu_1)^{-1}\,(\lambda_2/\mu_1)^{N-r-1}\,q\,(r+1; 1) + (\lambda_1/\mu_1)\,\{\lambda_2/(\lambda_2 + \mu_2) \\ & -\lambda_2/\mu_1\}^{-1}\,\{\{\lambda_2/(\lambda_2 + \mu_2)\}^{N-R-1} - (\lambda_2/\mu_1)^{N-R-1}\,\{\lambda_2/(\lambda_2 + \mu_2) - \lambda_2/\mu_1 \\ & + \lambda_2^2\,\{\mu_1\,(\lambda_2 + \mu_2)\}^{-1}\}\,p\,(R-1; 2)] + \mu_1\, z^{N-1}\,[\{1 - (\lambda_2\mu_1)^{R-r}\} \\ & \times (1-\lambda_2/\mu_1)^{-1}\,(\lambda_2/\mu_1)^{N-R}\,q\,(r+1; 1) \\ & + \lambda_1\,(\lambda_2 + \mu_2)\,\{\mu_1\,(\mu_1 - \lambda_2 - \mu_2)\}^{-1}\,\{\{\lambda_2/(\lambda_2 + \mu_2)\}^{N-R+1} \\ & - (\lambda_2/\mu_1)^{N-R-1}\,\lambda_2\,(\mu_1 - \mu_2)\,\{\mu_1\,(\lambda_2 + \mu_2)\}^{-1}\,p\,(R-1; 2)\}] \qquad \ldots(4.8) \end{aligned}$$

$$\delta_2\,(z) = \lambda_2\,(1-z) + \mu_1\,(1-z^{-1}) \qquad \ldots(4.9)$$

$$\delta_3(z) = \lambda_1 z^R p(R-1; 2) - \lambda_1 \{\lambda_2/(\lambda_2 + \mu_2)\}^{N-R} z^N p(R-1: 2) \quad \ldots(4.10)$$

$$\delta_4(z) = \lambda_2(1-z) + \mu_2.$$

The unknown quantities in $\Delta_i(z)$ $(i = 1, 2)$ and $\delta_j(z)$ $(j = 1, 2)$ are $p(R-1; 1)$, $p(R-1; 2)$, $q(r+1; 1)$ and $p(0)$, and these satiffy

$$\left.\begin{aligned} &\lambda_1 p(R-1; 1) + \lambda_1 p(R-1; 2) - \mu_1 q(r+1; 0) = 0 \\ &(z_i - 1) p(0) + (z_i^r - z_i^R) p(R-1; 1) \\ &+ \{z_i^r + \mu_2 z_i^{R-1}/(\lambda_1 z_i - \lambda_1 - \mu_2)\} p(R-1; 2) = 0 \; (i = 1,2) \end{aligned}\right\} \quad \ldots(4.11)$$

and the equation that the sum of all the probabilities is unity. Note that $\Delta(z)$ given by (4.7) has three zeros given by $z = 1$ and $z = z_i$ $(i = 1,2)$, (say).

The expressions for $P(z; i)$ $(i = 1, 2)$ and eqns. (4.11) coincide with those obtained by Scott and Hsia[2]. They however did not use the p. g. f. $Q(z; 1)$ and $Q(z; 2)$ as they derived $q(n; 1)$ $(n = r+1, r+2, \ldots N-1)$ and $q(n; 2)$ $(n = R, R+1, \ldots N-1)$ directly from the corresponding steady state equations for $q(n; 1)$ and $q(n; 2)$. We have verified that $q(n; 1)$, $q(n; 2)$ obtained from $Q(z: 1)$ and $Q(z; 2)$ respectively here coincide with those obtained by Scott and Hsia[2].

Now we find the expected number of customers in the system at time t as

$$L(t) = \sum_{j=1}^{4} L_j(t)$$

where

$$L_i(t) = \sum_{n=1}^{R-1} n\, p(t, n; i), \; (i = 1, 2)$$

$$L_3(t) = \sum_{n=r+1}^{N-1} n\, q(t, n; 1) + N\, q(t, N; 1)$$

$$L_4(t) = \sum_{n=R}^{N-1} n\, q(t, n; 2) + N\, q(t, N; 2).$$

Taking the L. T. w. r. to t we can now determine $\bar{L}(s)$ as

$$\bar{L}(s) = \sum_{j=1}^{4} \bar{L}_j(s)$$

where $\bar{L}_j(s)$'s are expressed in terms of partial derivatives of L. T.'s of the p. g. f. $\bar{P}(s, z; i)$ and $\bar{Q}(s, z; i)$ $(i = 1, 2)$ w. r. to z at $z = 1$. Using the property (4.1) the expected system size for the steady state case is given by

$$L = \sum_{j=1}^{4} L_j$$

where

$$L_j = \lim_{s \to 0} s\, L_j(s) \; (j = 1, 2, 3; 4).$$

Now,

$$L_1 = \lim_{z \to 1} \frac{\partial P(z; 1)}{\partial z}$$

$$= \{\mu_1 \mu_2 - \lambda_1 (\mu_1 + \mu_2)\}^{-1} [\lambda_1 (\mu_2 - \lambda_1)\, p(0) + \frac{\lambda_1}{2} \{2(R+1)\lambda_1 - R(R+1)\mu_2\}\, p(R-1;1) - \frac{\lambda_1}{2} \mu_2 R(R-1)\, p(R-1; 2) + \frac{\mu_1}{2} \{\mu_2 r(r+1) - 2(r+1)\lambda_1\}\, q(r+1; 1) + 2\lambda_1 (\lambda_1 - \mu_1 - \mu_2) P_1]$$

where

$$P_1 = \sum_{n=1}^{R-1} p(n; 1)$$

$$= \{\mu_1 \mu_2 - \lambda_1 (\mu_1 + \mu_2)\}^{-1} [\lambda_1 \mu_2\, p(0) + \lambda_1 (\mu_2 - \mu_1)\, p(R-1; 2) - (R-r)\, q(r+1; 1)].$$

Similarly,

$$L_2 = \lim_{z \to 1} \frac{\partial P(z; 2)}{\partial z}$$

$$= \{\mu_1 \mu_2 - \lambda_1 (\mu_1 + \mu_2)\}^{-1} [\lambda_1^2 (R+1)\, p(R-1;2) + \frac{1}{2} \lambda_1 \mu_1\, p(R-1;2) - \lambda_1 \mu_1\, p(0) - \frac{1}{2} \lambda_1 \mu_1 R(R+1)\, p(R-1; 1) + \frac{1}{2} \mu_1^2\, r(r+1)\, q(r+1; 1) - 2\lambda_1 (\lambda_1 - \mu_1 - \mu_2) P_2]$$

where

$$P_2 = \{\mu_1 \mu_2 - \lambda_1 (\mu_1 + \mu_2)\}^{-1} [\lambda_1^2\, p(R-1; 2) - \lambda_1 \mu_1 p(R-1; 2) + \lambda_1 \mu_1 p(0) - \lambda_1 \mu_1 (R+1)\, p(R-1; 1) + \mu_1^2 (r+1)\, q(r+1; 1)];$$

$$L_3 = \lim_{z\to 1} \frac{\partial Q(z;1)}{\partial z} + N\,q(N;1)$$

$$= (\mu_1-\lambda_2)^{-1}\left[\frac{1}{2}\mu_1(R-r)(R+r+1)\,q(r+1;1) + \lambda_1\{\lambda_2(R\mu_2 + \lambda_2)/\mu_2^2 - R - \lambda_2(N\mu_2+\lambda_2)/\mu_2^2\{\lambda_2/(\lambda_2+\mu_2)\}^{N-R}\}\,p(R-1;2) + \lambda_2 P_3 - \lambda_2(N+1)\,q(N;1)\right]$$

where

$$P_3 = \sum_{r+1}^{N} q(n;1)$$

$$= (\mu_1-\lambda_2)^{-1}[\mu_1(R-r)\,q(r+1;1) - (R+1)\lambda_1 p(R-1;2) - \lambda_2 q(N-1;1) + \lambda_1\mu_2^{-1}\{\lambda_2 + R\mu_2 - \lambda_2\{\lambda_2/(\lambda_2+\mu_2)\}^{N-K}\} p(R-1;2)] + q(N;1);$$

and

$$L_4 = \lim_{z\to 1} \frac{\partial Q(z;2)}{\partial z}$$

$$= \lambda_1\mu_2^{-2}(R\mu_2 + \lambda_2\{1 - \{\lambda_2/(\lambda_2+\mu_2)\}^{N-R}\})\,p(R-1;2).$$

These results coincide with those given by Scott and Hsia[2].

5. Conclusion

The techniques of p. g. f. and L. T. are applied to solve the queueing model considered here. The L. T.'s (in time) of the different p. g. f's and the average system size are derived and from these the steady state results are readily deduced. The inversion of the L. T.'s seems to be complicated and its evaluation is not attempted here.

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