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THE KLEIN IMAGE OF A LINE COMPLEX WHICH ADMIT A STRATIFICATION OF A NON *W*-CONGRUENCES

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This work concerned mainly with investigation and construction of the Klein images of line complexes immersed in S_3, on the absolutum of the space S_5. These line complexes admit a stratification into a one-parametric family of line congruences which are not W-congruences. The Klein images of these complexes and their related congruences are constructed and studied. Also the inverse K-images are established.

1. Introduction

We consider a 3-dimensional elliptic space S_3 referred to a moving frame as an orthonormal polar-tetrahedron $T(A_0, A_1, A_2, A_3)$ of four linearly independent analytic A_k $(k = 0, 1, 2, 3)$. An infinitesimal displacement of such a frame is determined by the matrix equation.

$$dA = \omega A \qquad \text{.. (1)}$$

where $A = (A_0\, A_1\, A_2\, A_3)'$ is column matrix and $\omega = (\omega_k^j)$ $(k, j = 0, 1, 2, 3)$ is a skew-symmetric square matrix of rank four, which is called Cartan's matrix corresponding to the elliptic space S_3 (Soliman *et al.*[3]). The elements ω_k^j are the invariant one forms (Pfaff's differential forms) of the elliptic group of transformation, whose structural equations have the form

$$D\omega_i^j = \omega_i^{\alpha} \Lambda\, \omega^j - \omega_0^j \Lambda\, \omega_0^j, \; (\alpha, i = 0, 1, 2, 3) \qquad \text{...(2)}$$

where D denotes the exterior differential operator and Λ denotes the exterior product between the differential forms.

The Klein mapping $K : Gr(1, 3) \rightarrow P^4 \subset S_5$, from the set of lines $Gr(1; 3)$ (Grassmann manifold) of the space S_3 onto the points of four dimensional Klein hyperquadric P^4 considered as the absolutum of the five-dimensional elliptic space S_5

(K-quadric). It is well-known[4] that the Grassman manifold $Gr\ (1;\ 3)$ has dimension four. The K-mapping maps the set of all lines $Gr\ (1;\ 3)$ bijectively onto the points of the K-quadric P^4 in the Klein five-dimensional elliptic space (K-space). In virtue of this definition the line $A_i\ A_j$ corresponds a point $l = [A_i,\ A_j]$ which is called the Klein point (K-point) belongs to $P^4 \subset S_5$, where $[A_i,\ A_j]$ denotes the Grassman product of the two points A_i, A_j (the vertices of $T\,(A_0,\ A_1,\ A_2,\ A_3)$). In this case l is determined by the six-tuples $(l_1,\ l_2, \ldots,\ l_6)$ and satisfies the Plücker condition $\Omega\ (l,\ l) = l_1\ l_4 + l_2\ l_5 + l_3\ l_6 = 0$. From these definitions, it follows that the line manifold immersed in S_3 can be studied as a point submanifold on $P^4 \subset S_5$ (Hlavaty[4]).

In order to study, the geometrical properties of the K-images of line manifolds immersed in S_3 on the K-quadric P^4, it is necessary to construct the moving frame attached to any point $P^4 \subset S_5$ (Kovansov[5]). We take the rectangle normalized hexahedron $H\ (P_i;\ \bar{P}_i)$, $i = 1, 2, 3$ formed by the six edges of the frame T conjugate to any K-point on P^4. The K-points P_i, $\bar{P}_i$ satisfying the following conditions

$$\Omega\ (P_i,\ \bar{P}_j) = \delta_{ij},\ < P_i,\ \bar{P}_j > = 0,\ j = 1, 2, 3$$

$$< P_i,\ P_j > = \delta_{ij},\ < \bar{P}_i,\ \bar{P}_j > = \delta_{ij}$$

where $P_i = [A_0,\ A_i]$, $\bar{P}_j$ $(i \neq j)$ $\bar{P}_i$ are its polar conjugate with respect to the absolutum of S_5 and $<\ ,\ >$ denotes the scalar product in S_5.

The infinitesimal displacement of the frame $H\ (P_i;\ \bar{P}_i)$ is given by

$$\begin{bmatrix} dp \\ d\bar{P} \end{bmatrix} = \begin{bmatrix} \phi & \theta \\ \theta & \phi \end{bmatrix} \begin{bmatrix} P \\ \bar{P} \end{bmatrix} \qquad \ldots(3)$$

where $\phi = (\phi_{ij})$, $\theta = (\theta_{ij})$ are skew-symmetric matrices of order three with the following $\phi_{12} = \omega_1^2$, $\phi_{13} = \omega_1^3$, $\phi_{23} = \omega_2^3$, $\theta_{12} = \omega_0^3$, $\theta_{13} = \omega_0^2$, $\theta_{23} = \omega_0^1$ and P, $\bar{P}$ are column vectors such that $P = (P_i)$, $\bar{P} = (\bar{P}_i)$ (Soliman *et al.*[3]).

2. Line Complexes Immersed in the Space S_3

We take the generator l of the Grassman manifold $Gr\ (1;\ 3)$ as the edge $A_0\ A_3$ of the frame $T\ (A_0,\ A_1,\ A_2,\ A_3)$. The invariance conditions of the line $l = A_0\ A_3$ under the transformation group of S_3 are $\omega_\gamma^\nu = 0$, $\gamma = 0, 3$; $\nu = 1, 2$. The line complex defined as a 3-dimensional submanifold of $Gr\ (1,3)$ and is denoted by $Gr\ (1,\ 3;\ 3)$ i.e., any line of $Gr\ (1,\ 3;\ 3)$ depends only upon three principal forms say ω_0^2, ω_0^1, ω_3^2. The differential equations which characterize this line complex related to a canonical moving frame (the planes $A_0\ A_3\ A_1$, $A_0\ A_3\ A_2$ are coincident with the planes correspond to the points A_0, A_3 in the normal correlation respectively) are given by

$$\begin{bmatrix} \omega_3^1 \\ d\chi \\ \omega_0^3 - \chi\omega_1^2 \\ \chi\omega_0^3 - \omega_1^2 \end{bmatrix} = \begin{bmatrix} \chi & 0 & 0 \\ a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} \omega_0^2 \\ \omega_0^1 \\ \\ \omega_3^2 \end{bmatrix}, \quad 1 - \chi^2 \neq 0 \qquad \ldots(4)$$

where χ (the curvature of the line complex) and (a_{ij}) are invariants in the first and second order contact elements of the ray l of the line complex respectively[5].

A one-dimensional submanifold of $Gr\,(1, 3)$ is called a ruled surface and is denoted by $Gr\,(1, 3; 1)$. The ruled surface $Gr\,(1, 3; 1)$ of the line complex $Gr\,(1, 3; 3) \subset Gr\,(1, 3)$ is called a principal surface if the osculating lines are coincident with the asymptotic lines[5]. A ruled surface $Gr\,(1, 3; 1)$ is called a coordinate surface of the line complex $Gr\,(1, 3; 3)$ if the centres A_0, A_3 of the ray, $l = A_0 A_3$ coincides with the osculating points. This coordinate surface is given by the Pfaff's differential equations

$$\omega_0^1 = 0, \ \omega_3^2 = 0. \qquad \ldots(5)$$

Now, we consider the line complex (4) for which one of the principal surfaces coincident with the coordinate surface (5). This geometrical property gives, the condition $a_{12} = a_{13} = 0$. The integrability conditions of the system (4) with the condition $a_{12} = a_{13} = 0$, give the following necessary condition $a_{11}\,(a_{22} - a_{33}) = 0$. Thus, we have two classes of line complexes having the above geometrical property, which are given by

$$a_{12} = a_{13} = a_{11}, \quad a_{22} \neq a_{33} \qquad \ldots(6)$$

$$a_{12} = a_{13} = 0, \ a_{22} = a_{33}, a_{11} \neq 0. \qquad \ldots(7)$$

The existence theorems of these classes of line complexes are given and their integral free-representation are obtained by Soliman and Abdel-All[6,7].

3. The K-images of Line Complexes on the Absolutum of the K-Space S_5

It is well-known[3,4,8] that the K-image of a line complex is a 3-dimensional point manifold (3-dimensional surface) on the absolutum of the K-space S_5. We denote the K-image of the line complex (4) by $\sigma_3 \equiv (P_3)$ which described by the vertex P_3 of the frame $H\,(P_i; \bar{P}_i)$ of S_5 where

$$dP_3 = \omega_0^2\,(\bar{P}_1 + \chi P_1), \omega_3^2\, P_2 - \omega_0^1\, \bar{P}_2. \qquad \ldots(8)$$

From (8), it follows that the 3-dimensional tangent K-space of the line complex (4) and its K-image σ_3 is the K-space whose tangential coordinates is $(P_3, P_2, \bar{P}_2, \bar{P}_1 + \chi P_1)$. The second osculating K-space denotes by $T^2_{P_3}\,(\sigma_3)$ of σ_3 is obtained from[9].

$$d^2 P_3 \equiv \{\omega_2^3 (\omega_1^2 - \chi\omega_0^3) + \omega_0^1 (\omega_0^3 - \chi\omega_1^2) + \omega_0^2 \, d\chi\} P_1$$
$$- (\omega_0^1 \omega_2^3 + \chi (\omega_0^2)^2) \bar{P}_3 \; (\text{mod } P_3, P_2, \bar{P}_2, \bar{P}_1 + \chi P_1). \quad ..(9)$$

From (8), it follows that the normal space, denoted by $N_{P_3}(\sigma_3)$, contains the K-points P_1 and $\bar{P}_3$. Therefore, the first and the two 2nd fundamental forms of σ_3 denoted by I and II_1, II_2 are given as[3]

$$I \equiv \langle dP_3, dP_3 \rangle, \; II_1 \equiv \langle d^2 P_3, P_1 \rangle, \; II_2 \equiv \langle d^2 P_3, \bar{P}_3 \rangle \quad ...(10)$$

respectively[3]. The Gaussian curvature K_3 and the mean curvature vector $\mathbf{H}_3$ are defined by[4,10].

$$\left.\begin{aligned} K_3 &= \frac{\text{Det } II_1 + \text{Det } II_2}{\text{Det } I} + 1 \\ \mathbf{H}_3 &= (\text{trac } II_1) P_1 + (\text{trac } II_2) \bar{P}_3. \end{aligned}\right\} \quad ...(11)$$

As a continuation of the previous papers[11-13] here, we study the K-images of line complexes for which $a_{22} - a_{33} = 0$, $a_{11} \neq 0$ and $a_{22} = \chi \, a_{23}$. The integrability conditions of the system of equations (4) with the above conditions imply the condition $a_{23} = \pm \, i \; (i = \sqrt{-1})$. Thus, without loss of generality the line complexes under investigation are determined by the following system of differential equations

$$\left.\begin{aligned} \omega_3^1 &= \chi\omega_0^2, \; d\chi = a_{11} \, \omega_0^2 \\ \omega_0^3 &= i\omega_0^1, \; \omega_1^2 = - i\omega_3^2. \end{aligned}\right\} \quad ...(12)$$

The range of existence of the line complexes (12) comprises one arbitrary function of one argument[13].

We denote the K-image of a line complex of the system (12) by σ_3^2.

From (12), (8), (9), (10) we get the following fundamental quadratic forms :

$$\left.\begin{aligned} I &= (\omega_0^1)^2 + (\omega_3^2)^2 + (1 + \chi^2)(\omega_2^3)^2 \\ II_1 &= i(\omega_0^1)^2 + i(\omega_3^2)^2 + a_{11}(\omega_0^2)^2 + 2\chi \, i\omega_0^1 \, \omega_3^2 \\ II_2 &= 2\omega_0^1 \, \omega_2^2 - 2\chi(\omega_0^2)^2. \end{aligned}\right\} \quad ...(13)$$

From (13), (11) one can easily see that the Gaussian curuature denoted by K_3^2 and the mean curvature vector denoted by $\mathbf{H}_3^2$ of the K-image σ_3^2 are given by,

$$\left.\begin{aligned} K_3^2 &= ((1 + \chi)^2 + a_{11}(\chi^2 - 1))/(1 + \chi^2) \\ \mathbf{H}_3^2 &= ((a_{11}/(1 + \chi^2)) + 2i) P_1 - (2\chi/(1 + \chi^2)) \bar{P}_5 \end{aligned}\right\} \quad ...(14)$$

respectively. The line complex (12) having the property that, the line $A_0 A_2$ generates a plane tangent to the absolutum of the space S_5 at the fixed point $A_1 + iA_3$. This plane has the tangential coordinates ($A_0 A_2$, $A_1 + iA_3$). The K-image of the line $A_0 A_2$ is the K-point P_2. From (3), (12) we have the following differentials :

$$dP_2 = \omega_3^2 (iP_1 - P_3) + \omega_0^1 (\bar{P}_3 - i\bar{P}_1)$$

$$d^2 P_2 \equiv 0 \,(\text{mod } P_2, iP_1 - P_3, \bar{P}_3 - i\bar{P}_1)\,.$$

From these differentials, it is easy to see that the point P_2 describes a 2-dimensional surface on $P^4 \subset S_5$ with a fixed 2nd osculating space. So the 2-dimensional surface degenerates into a 2-dimensional plane whose tangential coordinates (P_2, $iP_1 - P_3$, $\bar{P}_3 - i\bar{P}_1$).

Thus, we have the following theorem :

Theorem 1—When the ray $A_0 A_3$ generates the line complex (12) in S_3, then $A_0 A_2$ generates (a congruence) a family of lines in a fixed plane, tangent to the absolutum of S_3 at the point $A_1 + iA_3$ (Hlavaty[4]). The K-image of this family is a 2-dimensional K-plane immersed in the absolutum of the K-space S_5.

Now, we investigate the K-inverse image of a 3-dimensional surface immersed in S_5, for which the Otsuki's curvature corresponding to the fundamental form II_2 equals zero Rosca *et al.*[14] From (14) we get $\chi = 0$, $a_{11} = 0$ and $K_3^3 = 1$, $\mathbf{H}_3^3 = 2\, i\, P_1$ which characterize a generalized W-surface. Thus the K-inverse image of this generalized W-surface is the line complex which is determined by the following equations :

$$\omega_3^1 = 0,\ \omega^3 = i\,\omega^1,\ \omega_1^2 = -\,i\,\omega_3^2,\ i = \sqrt{-1}. \qquad \text{.. (15)}$$

The line complex (15) exists within three constants, and constructed as the set of all tangent lines to a 2-dimensional surface described by the vertex A_3 of the frame $T(A_0, A_1, A_2, A_3)$ in S_3. This surface has the normal vector $A_3 A_1$ and from the differentials.

$$\left.\begin{aligned} dA_3 &= -\,\omega_0^3\, A_0 + \omega_3^2\, A_2 \\ d^3 A_3 &\equiv -\,(\omega_0^1\, \omega_0^3 + \omega_3^2\, \omega_1^2)\, A_1 \,(\text{mod } A_3, A_0, A_2) \end{aligned}\right\}. \qquad \text{...(16)}$$

Then, one can easily obtain that the Gaussian and mean curvature assume the values zero and i respectively, which characterizes a Clifford surface immersed in S_3 (Kovansov[5]).

Thus, we have the following theorem :

Theorem 2—The K-inverse image of a 3-dimensional surface immersed in the K-space S_5, for which one of Otouki's curvature corresponding to the 2nd-fundamental form II_2 vanishes identically is a special line complex. This line complex constructed as the family of all tangent lines to a certain Clifford surface.

4. The K-images of Line Congruences related to Line Complexes

Soliman and Abdel-All[13] proved that the line complex (12) can be stratified into a one-parametric family of holonomic line congruences determined by the following differential equations :

$$\left.\begin{array}{l}\omega_0^2 = 0, \quad d\chi = a_{11}\,\omega_0^2 \\ \omega_0^3 = i\omega_0^1, \; \omega_1^2 = -\,i\,\omega_3^2\end{array}\right\}. \qquad \ldots(17)$$

The line congruence (17) has two focal surfaces, one of them is degenerate to a fixed line its K-image is the K-point $P_1 + iP_3 \in P^4$ and the other is a horosphere with centre on the absolutum of the space S_3.

The K-image of this line congruence is a K-surface immersed in $P^4 \subset S_5$ for this K-surface we have the differentials :

$$\left.\begin{array}{l}dP_3 = \omega_3^2\, P_2 - \omega_0^1\, \bar{P}_2 \\ d^2 P^3 \equiv (i\,(\omega_0)^2 + i\,(\omega_3^1)^2)\, P_1 + 2\,\omega_0^1\,\omega_3^2\,(\bar{P}_3 - i\bar{P}_1) \\ \qquad (\text{mod } P_3, P_2, \bar{P}_2).\end{array}\right\} \qquad \ldots(18)$$

From (18), it follows that the Gaussian curvature assume the value -4 and the mean curvature vector is $-iP_1$. Thus we have proved the following theorem.

Theorem 3—The K-image of the line congruence (17) is a K-surface with negative Gaussian curvature assuming the value -4 and constant mean curvature which assumes the value -1.

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