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TRIANGLE CONTRACTIVE MAPS AND THEIR FIXTURES IN NORMED LINEAR SPACES

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The area of a triangle in a Hilbert space equals half the product of a side and the perpendicular distance from the opposite vertex. In the setting of a normed linear space we define in two different ways the area of a triangle using the notion of orthogonality due to Birkhoff and James. In the first part of the paper we give a few results characterising inner product spaces amongst normed linear spaces using this notion of area of a triangle. In the second part we consider maps which contract triangles and obtain a few results regaıding their fixed points and fixed lines.

INTRODUCTION

The area of a triangle in a real Hilbert space has a clear meaning—the Euclidean area, which equals half the product of a side and the perpendicular distance from the opposite vertex, and which is given by a host of other formulae. We denote by $\Delta (x, y, z)$ the area of the triangle with vertices at $x, y, z \in H$. A map $f : H \to H$ is called triangle contractive (TC), if there exists a number α in $0 < \alpha < 1$ such that for any $x_1, x_2, x_3 \in H$ either

$$\Delta (f(x_1), f(x_2), f(x_3)) \leqslant \alpha \, \Delta (x_1, x_2, x_3)$$

or

$$\|f(x_i) - f(x_j)\| \leqslant \alpha \, \|x_i - x_j\|, \; i, \quad j = 1, 2, 3.$$

By a fixture of a TC map we mean a fixed point or a fixed line. Clearly any contraction is a TC map. TC maps in Hilbert spaces and their fixtures have been the subject of study in previous papers[1-4]. The object of the present paper is to extend the results meaningfully to the normed space setting.

In section 1 of this paper we define the notion of the area of a triangle in two different ways. The first way, for example, adopts the formula—half the product of the length of a side and the shortest distance of the opposite vertex from it. To obtain the shortest distance (or the perpendicular distance) of the opposite vertex from a side we use Birkhoff-James orthogonality. In general for a given triangle this product ($\frac{1}{2}$ base $\times$ height) depends upon the choice of a particular side as base. Hence we have taken the liberty of assigning a triplet of numbers as the area of a triangle, as against the usual assignment of just one number as the area. The situation is similar in the second method of defining the area of a triangle. A little closer look at the inter-relationship between the two types of areas leads us to some interesting characterizations of inner product spaces amongst normed linear spaces.

In section 2 we extend the notion of TC maps in normed linear spaces and make a study of the existence of their fixtures.

1. Area of Triangle

Throughout this paper X will be a real, strictly convex and smooth normed linear space. θ is the zero of X and $\hat{x}$ will denote $\frac{x}{\|x\|}$. The area of type I of a triangle with vertices x, y and z is defined to be the triplet

$$\Delta_1 (x, y, z) = \{A (x; y, z), A (y; z, x), A (z; x, y)\}$$

where

$$A (x; y, z) = \frac{1}{2} \|y-z\| \times \text{shortest distance of the vertex } x \text{ from the line through the points } y \text{ and } z$$

$$= \frac{1}{2} \|y-z\| \, \|x-(\lambda y + (1-\lambda) z)\|$$

where λ is the unique number for which $\|(x-(\lambda y + (1-\lambda) z)\|$ is minimum.

In terms of the notions of Birkhoff-James orthogonality[5] ($x \perp y \leftrightarrow \|x + \lambda y\| \geqslant \|x\|$ for all λ) and the normalized duality map J ($J : X \to X^*$ such that $\|J_x\| = \|x\|$, $(J_x, x) = \|x\|^2$) the following proposition is obvious.

Proposition 1.1—For any $x, y, z \in X$

(i) $A (x; y, z) = \frac{1}{2} \|y-z\| \, \|x-(\lambda y + (1-\lambda) z)\|$

where λ is the unique number such that

$$x-(\lambda y + (1-\lambda) z) \perp (y-z).$$

(ii) $A(x; y, z) = \frac{1}{2} \|y-z\| (J(x-(\lambda y + (1-\lambda) z)), x-z)^{1/2}$

where

$$(J(x-(\lambda y + (1-\lambda) z)), y-z) = 0.$$

(iii) $A(x; y, z) = A(x + w; y + w, z + w)$ for any $w \in X$.

(iv) $A(x; y, z)$ is a continuous function of x, y and z.

Using the well known fact that the orthogonality in a normed linear space X with dim $X \geqslant 3$ is symmetric only if the space is an inner product space[5], we prove the following :

Proposition 1.2—If dim $X \geqslant 3$ and $A(x; y, z) = A(y; z, x)$ for all $x, y, z \in X$, then X is an inner product space.

PROOF : Let $u, v \in X$ with $u \perp v$, then $A(u; v, \theta) = \frac{1}{2} \|v\| \|u\|$ (θ denotes the zero element of the space X). $A(v; \theta, u) = \frac{1}{2} \|u\| \|v-\lambda u\|$ where $v-\lambda u \perp u$.

$A(u; v, \theta) = A(v; \theta, u)$ implies $\|v\| = \|v-\lambda u\|$ and so $\|v\| \leqslant \|v + ku\|$ for all k, hence $v \perp u$. Thus the orthogonality is symmetric and consequently the space is an inner product space.

Area of type II of a triangle $x, y, z \in X$ is defined to be the triplet

$$\Delta_2(x, y, z) = \{B(x; y, z), B(y; z, x)\ B(z; x, y)\}$$

where

$$B(x; y, z) = \frac{1}{2} \|y-z\| \|x-(ty + (1-t) z)\|$$

where t is the unique number (by smoothness) such that

$$y-z \perp x - (ty + (1-t) z).$$

Proposition 1.3—For $x, y, z \in X$

(i) $B(x; y, z) = \frac{1}{2} \|y-z\| \left\|x-z - \frac{(J'y-z), x-z)}{\|y-z\|^2}(y-z)\right\|$

(ii) $B(x; y, z) = B(x + w; y + w, z + w)$ for $w \in X$.

(iii) $B(x; y, z)$ is a continuous function of x, y and z.

Further

(iv) If dim $X \geqslant 3$ and for x, y, z, $B(x; y, z) = B(y; z, x)$, then the orthogonality is symmetric and hence the space is an inner product space.

PROOF : Easy.

We have observed in Propositions 2 and 3 that if $A(x; y, z) = A(y; z, x)$ or $B(x; y, z) = B(y; z, x)$ for all x, y and z, then orthogonality is symmetric. Conversely, if the dim $X \geqslant 3$ and orthogonality is symmetric then X is an inner product space, hence $A(x; y, z) = A(y; z, x)$ and $B(x; y, z) = B(y; z, x)$. If dim X is 2 and the orthogonality is symmetric, then we have the following :

Lemma 1.1—If dim $X = 2$ and orthogonality is symmetric then $B(x; y, \theta) = B(y; \theta, x)$ for any $x, y \in X$.

PROOF : Since $B(x; y, \theta) = \frac{1}{2} \|y\| \, \|x\| \, \|\hat{x} - (J\hat{y}, \hat{x}) \hat{y}\|$ and $B(y; \theta, x) = \frac{1}{2} \|x\| \, \|y\| \, \|\hat{y} - (J\hat{x}, \hat{y}) \hat{x}\|$, we have to prove that

$$\|\hat{x} - (J\hat{y}, \hat{x}) \hat{y}\| = \|\hat{y} - (J\hat{x}, \hat{y}) \hat{x}\|.$$

Introducing the coordinate system with respect to any two orthogonal vectors, as done by Day[6] we let

$$\hat{x} = (x_1, x_2), \hat{y} = (y_1, y_2) \text{ and } J_y = (p_y, q_y) \text{ so that}$$

$$p_y y_1 + q_y y_2 = 1 = \sup_{t, s \neq 0} \frac{| p_y t + q_y s |}{\|(t, s)\|}$$

$$\begin{aligned} \|x - (J_y, x) y\| &= \|(x_1, x_2) - (p_y x_1 + q_y x_2)(y_1 \, y_2)\| \\ &= \|(x_1 - p_y x_1 y_1 - q_y x_2 y_1, \, x_2 - p_y x_1 y_2 - q_y x_2 y_2)\| \\ &= \|(q_y y_2 x_1 - q_y x_2 y_1, \, p_y y_1 x_2 - p_y x_1 y_2)\| \\ &= | x_1 y_2 - x_2 y_1 | \, \|(q_y, -p_y)\|. \end{aligned}$$

But

$$(J_y, (q_y, -p_y)) = p_y q_y - p_y q_y = 0.$$

Therefore, we must have[6], that

$$1 = | p_y y_1 + q_y y_2 | = \|(y_1, y_2)\| \, \|(q_y, -p_y)\| = \|(q_y, -p_y)\|.$$

Thus

$$\|\hat{x} - (\hat{J_y}, \hat{x}) \hat{y}\| = | x_1 y_2 - x_2 y_1 | = \|\hat{y} - (J\hat{x}, \hat{y}) \hat{x}\|.$$

This completes the proof.

Theorem 1.1—The orthogonality in a normed linear space is symmetric if and only if either $A(x; y, z) = A(y; z, x)$ or $B(x; y, z) = B(y; z, x)$ for all $x, y, z \in X$.

PROOF : If the orthogonality is symmetric then $A(x; y, z) = B(x; y, z)$ and $A(y; z, x) = B(y; z, x)$. Also by *Lemma* 1.1 $B(x; y, z) = B(x - z; y-z, \theta) = B(y-z; \theta, x-z) = B(y; z, x)$. Hence the result.

Theorem 1.2—If in a two dimensional space X the orthogonality is symmetric then

$$\Delta_1(x, y, z) = \Delta_2(x, y, z) = \frac{1}{2} \mid (y z_2 - y_2 z_1) + (z_1 x_2 - z_2 x_1) + (x_1 y_2 - x_2 y_1) \mid$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$ with respect to two orthogonal unit vectors.

PROOF : From Lemma 1.1 and Theorem 1.1, it follows that $\Delta_1(x, y, z) = \Delta_2(x, y, z) = B(x, y, z)$. Also from the proof of Lemma 1 we have

$$\begin{aligned} B(x; y, z) &= B(x-z; y-z, \theta) \\ &= \frac{1}{2} \mid (x_1-z_1)(y_2-z_2) - (x_2-z_2)(y_1-z_1) \mid \\ &= \frac{1}{2} \mid (y_1 z_1 - y_2 z_1) + (z_1 x_2 - z_2 x_1) + (x_1 y_2 - x_2 y_1) \mid . \end{aligned}$$

Remark : The result of Theorem 1.2 is very interesting in the sense that here we get the Euclidean formula for the area of a triangle in a plane in terms of the coordinates of its vertices.

Before taking up the study of triangle contractive maps, we prove the following

Lemma 1.2—If x and y are linearly independent vectors, $\epsilon > 0$, then there exist numbers $a > 0$ and $b > 0$ such that

(i) $A(z; \theta, y) + A(z; x, \theta) \geqslant a$, and

(ii) $B(z; \theta, y) + B(z; x, \theta) \geqslant b$

for all $z \in X$ with $\|z\| \geqslant \epsilon$.

PROOF : (i) Let $A(z; \theta, y) = \frac{1}{2}\|y\| \|z-ty\|$ and $A(z; x, \theta) = \frac{1}{2}\|x\| \|z-sx\|$ where t and s are such that $z-ty \perp y$ and $z-sx \perp x$.

$$A(z; \theta, y) + A(z; x, \theta) \geqslant \frac{1}{2} \min \{\|x\|, \|y\|\} [\|z-ty\| + \|z-sx\|].$$

We claim that $\inf [\|z-ty\| + \|z-sx\| : \|z\| \geqslant \epsilon] > 0$. If not, there exist sequences $\{t_n\}$, $\{s_n\}$ and $\{z_n\}$ such that $z_n - t_n y \perp y$ and $z_n - s_n x \perp x$ and $\|z_n - t_n y\| + \|z_n - s_n x\| \to 0$. This means $z_n - t_n y \to 0$ and $z_n - s_n x \to 0$. Hence $t_n y - s_n x \to 0$, which is not possible in view of the linear independence of x and y unless $t_n \to 0$ and $s_n \to 0$, but then $z_n \to 0$ which contradicts the hypothesis that $\|z_n\| \geqslant \epsilon$. Now set

$$a = \frac{\inf\{\|z-ty\| + \|z-sx\| : \|z\| \geqslant \epsilon,\ z-ty \perp y,\ z-sx \perp x\}}{\frac{1}{2}\min\{\|x\|, \|y\|\}}.$$

For (ii) a similar proof can be provided.

2. Triangle Contractive Maps and their Fixtures

Let $f : X \to X$; we say

(i) f is triangle contractive of type I (TC I) if for some α, $0 < \alpha < 1$, and for any triangle with vertices x, y, z either

$$\|fx - fy\| \leqslant \alpha \|x-y\|, \|fy - fz\| \leqslant \alpha \|y-z\|$$

and

$$\|fz - fx\| \leqslant \alpha \|z-x\|$$

or

$$A(fx; fy, fz) \leqslant \alpha A(x; y, z),\ A(fy; fz, fx) \leqslant \alpha A(y; z, x)$$

$$A(fz; fx, fy) \leqslant \alpha A(z; x, y).$$

(ii) f is triangle contractive of type II (TC II) if (i) holds with A replaced by B.

(iii) f is tringle contractive (TC) if f is TC I or TC II. We say that $f : X \to X$ is bounded if it takes bounded sets into bounded sets. A fixture of map $f : X \to X$ is defined to be either a point p such that $f(p) = p$ or a line L such that $f(L) \subset L$.

Our next theorem asserts that we can expect to meet few triangle contractive maps which are discontinuous or which are unbounded. In any case a TC map which is either discontinuous or unbounded must have a fixture.

Theorem 2.1—If f is TC I or TC II and is either not continuous or not bounded then $f(X)$ is contained in a line.

Proof: (a) Suppose f is TC I, and is not continuous at x. Choose a sequence $\{x_n\}$ and $\epsilon > 0$ such that $x_n \to x$ but $\|fx_n - fx\| \geqslant \epsilon$ for all n. Consider the L through $f(x_1)$ and $f(x)$ i.e. the line $L = \{tf(x) + (1-t)f(x_1) : t \in R\}$. If $f(X) \not\subset L$, choose $z \in X$ such that $f(z) \notin L$.

Since $\|fx_n - fx\| > \alpha \|x_n - x\|$ for all sufficiently large n, we must have

$$A\,(fx_n; fx, fx_1) \leqslant \alpha A\,(x_n; x, x_1)$$

and

$$A\,(fx_n; fx, fz) \leqslant \alpha A\,(x_n; x, z).$$

By Proposition 1.1

$$A\,(x_n; x, x_1) = A\,(x_n - x; \theta, x_1 - x) \to 0 \text{ as } n \to \infty.$$

Similarly

$$A\,(x_n; x, z) \to 0 \text{ as } n \to \infty.$$

On the other hand, since $fx_1 - fx$ and $f(z) - f(x)$ are linearly independent, by Lemma 1.2

$$A\,(fx_n; fx, fx_1) + A\,(fx_n; fx; fz) \geqslant a > 0.$$

This leads to a contradiction, hence $f(X) \subset L$.

(ii) Suppose f is TC I and not bounded. Let $\{x_n\}$ be a bounded sequence such that $\|f(x_n)\| \to \infty$. Suppose x and $y \in X$ such that $f(x) \neq f(y)$. Since for all sufficiently large n

$$\|f(x_n) - f(x)\| > \alpha \|x_n - x\|$$

therefore for all sufficiently large n

$$A\,(f(y); f(x_n), f(x)) \leqslant \alpha\, A\,(y; x_n, x).$$

Thus

$$\|f(x_n) - f(x)\|\, \|(f(y) - f(x)) + t_n\,(f(x_n) - f(x))\| \leqslant \alpha\, \|x_n - x\|\, \|y - x + s_n\,(x_n - x)\|$$

where t_n and s_n are such that

$$f(y) - f(x) + t_n\,(f(x_n) - f(x)) \perp f(x_n) - f(x),\ y - x + s_n\,(x_n - x) \perp x_n - x.$$

It follows from the above that for all sufficiently large n

$$\|f(y) - f(x) + t_n\,(f(x_n) - f(x))\| \leqslant \frac{\alpha\,\|x_n - x\|}{\|f(x_n) - f(x)\|}\,\|y - x\|.$$

This yields that

$$\|f(y) - f(x) + t_n (f(x_n) - f(x))\| \to 0.$$

From here we get that $\{f(x_n) - f(x)\}$ converges to an element w, if need be we can pass on to a subsequence, and that $f(y) = f(x) + tw$, which gives the desired result.

In the same way we can prove the result for maps which are TC II.

Before proceeding further we introduce the following property :

A map $f : X \to X$ is said to have the property (a) if "there exists a fixed point of f or there exists a sequence $\{x_n\}$ in X and sequence $\{t_n\}$ of numbers such that $f(x_n) = t_n x_n$, $t_n > 1$ for all n, $\|x_n\| \to \infty$ and $\{\hat{x}_n\}$ converges".

A continuous map on a finite-dimensional space has the property (a). This is an easy consequence of the Brouwer fixed point theorem. We shall show in Theorem 3.6 below that there are infinite dimensional situations also in which TC maps have the property (a).

Theorem 2.2—Let $f : X \to X$ be a TC map with property (a). Then either f has a fixed point or there is a direction such that lines parallel to this direction are mapped on lines parallel to it.

PROOF : *Case* 1—f is TC I. For any $\beta \in (\alpha, 1)$ and $x \in X$, we have for all sufficiently large n

$$\|fx_n - fx\| > \beta \|x_n - x\| \qquad \ldots(1)$$

where $\{x_n\}$ is a sequence in property (a).

Hence $A(fy; fx_n, fx) \leqslant \alpha A(y; x_n, x)$ for any $y \in X$, i.e.

$$\|fx_n - fx\| \|fy - fx + s_n (fx - fx_n)\|$$

$$\leqslant \alpha \|x_n - x\| \|y - x + s_n' (x - x_n)\|.$$

Since

$$\frac{\|x_n - x\|}{\|fx_n - fx\|} \leqslant \frac{1}{\beta}$$

for sufficiently large n, for any $\beta \in (\alpha, 1)$, we have

$$\|fy - fx + s_n (fx - fx_n)\| \leqslant \frac{\alpha}{\beta} \|y - x + s_n' (x - x_n)\|. \qquad \ldots(2)$$

Here $fy-fx + s_n(fx-fx_n) \perp fx-fx_n$, hence

$$\|fy-fx + s_n(fx-fx_n)\| \leqslant \|fy-fx\|$$

and thus the sequence $\|fy-fx + s_n \|fx-fx_n\| \widehat{fx-fx_n}\|$ can be assumed to be convergent by passing to a subsequence. By hypothesis $\{\hat{x}_n\}$ is convergent to some w and hence $\{\widehat{x_n-x}\}$ and $\{\widehat{fx_n-fx}\}$ also converge to w. Taking the limit as $n \to \infty$, the inequality (2) yields

$$\|fy-fx + sw\| \leqslant \alpha \|y-x+s'w\| \qquad \ldots(3)$$

where $fy-fx+sw \perp w$ and $y-x+s'w \perp w$.

Now if x and y lie on a line parallel to w i.e. $y-x = tw$ then $y-x+s'w \perp w$ if and only if $y-x+s'w = 0$ and in this case $fy-fx = sw$, which shows that fx and fy also lie on a line parallel to w.

Case 2— Let f be TC II. The inequality (1) of the above proof implies $B(fx_n; fx, fy) \leqslant \alpha B(x_n; x, y)$ for all $y \in X$ and for all sufficiently large n. Thus

$$\|fx-fy\| \, \|fx_n-fx\| \, \|(\widehat{fx-fy}) - (J(\widehat{fx_n-fx}), \widehat{fx-fy})(\widehat{fx_n-fx})\| \leqslant \alpha \|x-y\| \, \|x_n-x\| \, \|\widehat{x-y} - (J(\widehat{x_n-x}), \widehat{x-y})(\widehat{x_n-x})\|. \qquad \ldots(4)$$

Since

$$\frac{\|x_n-x\|}{\|fx_n-fx\|} \leqslant \frac{1}{\beta}$$

for sufficiently large n, and any $\beta \in (\alpha, 1)$, taking the limit in (4) we get

$$\|fx-fy\| \, \|(\widehat{fx-fy}) - (Jw, \widehat{fx-fy})w\| \leqslant \alpha \|x-y\| \, \|(\widehat{x-y}) - (Jw, \widehat{x-y})w\|. \qquad \ldots(5)$$

From this inequality it follows that if x and y lie on a line parallel to w, then fx and fy also line on a line parallel to w.

Theorem 2.3—Let X be a Banach space, $f:X \to X$ a TC I map with property (a). Suppose that the orthogonality in X is left additive in the sense $x \perp z$ and $y \perp z$ implies $x+y \perp z$. Then f has a fixture.

PROOF : Either f has a fixed point or following the proof of Theorem 2.2, we get the inequality (3) as

$$\|fy-fx+sw\| \leqslant \alpha \|y-x+s'w\|$$

where $fy-fx+sw \perp w$ and $y-x+s'w \perp w$. From this we get

$$\|fy-fx+sw\| \leqslant \alpha \|y-x+s'w\| \leqslant \alpha \|y-x\|. \qquad \ldots(6)$$

We define $T: X \rightarrow X$ as follows $Tx = fx-tw$ where t is so chosen that $fx-tw \perp w$.

Now for any x and $y \in X$, let $Tx = fx-t_1w$ and $T_y = fy-t_2w$ then by left additivity of orthogonality $fx-fy-(t_1-t_2)w \perp w$. But then $s = t_2-t_1$, and by (6)

$$\|Tx-Ty\| = \|fx-fy+sw\| \leqslant \alpha \|x-y\|.$$

Therefore by Banach contraction mapping theorem T has a fixed point $x_0 \in X$ so that $Tx_0 = fx_0-t_0w = x_0$.

Now if L is the line through x_0 and fx_0 then it is parallel to w and it is mapped onto itself under f, by Theorem 2.2. Thus L is fixed under f.

Theorem 2.4—If X is a Banach space and $f: X \rightarrow X$ is a TC II maps with property (a), then f has a fixture.

PROOF : Either f has a fixed point or else we get the inequality (5) rewritten as

$$\|fx-fy-(Jw, fx-fy)\| \leqslant \alpha \|x-y-(Jw, x-y)w\|. \qquad \ldots(7)$$

Let

$$M = \{z \in X : w \perp z\} = \{z \in X : (Jw, z) = 0\}.$$

Then M is a closed hyperplane in X. We define $P_M : X \rightarrow M$, where for $x \in X$, $P_M(x) = x - (Jw, x)w$. From (7) we have for any $x, y \in M$

$$\|P_M fx - P_M fy\| \leqslant \alpha \|P_M x - P_M y\| = \alpha \|x-y\|.$$

This shows that $P_M f$ is a contraction on M. Let $x_0 \in M$ be a fixed point of $P_M f$. Let L be a line through x_0 parallel to w, then applying (7) to x_0 and any $u \in L$

$$\begin{aligned}\|P_M fu - P_M fx_0\| &\leqslant \alpha \|P_M u - x_0\| \\ &= \alpha \|u-(Jw, u)w - x_0\| \\ &= 0.\end{aligned}$$

Therefore $P_M fu = x_0 = fu-(Jw, fu)w$, which shows that $fu \in L$. Hence L is fixed under f.

Remark : In Theorem 2.3 we needed left additivity of orthogonality. It is not known which spaces other than inner product spaces will have this property. Two dimensional strictly convex spaces do have this property.

Theorem 2.5—Let X be finite-dimensional. $f: X \to X$ is a TC I map for which the following holds :

'There exists a bounded sequence $\{z_n\}$ in X such that $\|fz_n - z_n\| \to 0$ as $n \to \infty$'.
Then f has a fixture.

Proof : In view of Theorem 2.1 we can assume f is continuous. Therefore f has the property (a). Proceeding as in Theorem 2.3 we can obtain $w \in X$ such that the inequality (6) holds. Putting $y = fz_n$ and $x = z_n$ in (6) we get

$$\|f^2 z_n - fz_n - s_n w\| \leqslant \alpha \, \|fz_n - z_n - s_n' \, w\| \leqslant \alpha \, \|fz_n - z_n\|.$$

Since $\{z_n\}$ is a bounded sequence we can assume by passing to a subsequence if necessary that $z_n \to z_0$. The above inequality gives in the limit

$$\|f^2 z_0 - fz_0 - sw\| \leqslant \alpha \, \|fz_0 - z_0 - s' \, w\| \leqslant 0.$$

This implies that the line L through z_0 and fz_0 is parallel to w and is fixed.

Theorem 3.6—Suppose X is reflexive and $f: X \to X$.

(a) If f is completely continuous and TC I and $f\theta \neq fx$ for all $\|x\| \leqslant 1$, then f has the property (a).

(b) If f is TC II, completely continuous, $f(\theta) \neq f(x)$ for any $\|x\| \leqslant 1$, $\alpha < \frac{1}{2}$ and J is weak to weak continuous, then f has the property (a).

Proof : If f has no fixed point then by Schauder's fixed point theorem applied to the composite function $T_n f$ where T_n is the radial retraction on the ball $B_n = \{x : \|x\| \leqslant n\}$ we obtain a sequence $\{x_n\}$ such that $fx = t_n \, x_n$ with $t_n > 1$ and $\|x_n\| = n$. Suppose x_n converges weakly to w, passing on to a subsequence if need be. We will first show that $w \neq \theta$.

Case I—f is TC I, then by inequality (2) in Theorem 2.2, we have

$$\| f(y) - f(x) + s_n \, (fx - fx_n) \| \leqslant \frac{\alpha}{\beta} \, \|y - x\|$$

for all sufficiently large n and $\beta \in (\alpha, 1)$. By weak lower semicontinuity of the norm we obtain

$$\|fy - fx + sw\| \leqslant \alpha \, \|y - x\|.$$

If $w = \theta$, then f become a contraction, which is false (f has no fixed point).

Case II—If f is TC II then by inequality (4) of Theorem 2.2 we shall have

$$\|fx-fy\|\, \|(\widehat{fx-fy}) - (J(\widehat{fx_n-fx}), (\widehat{f(x)-f(y)}))(\widehat{fx_n-fx})\|$$

$$\leqslant \frac{\alpha}{\beta}\|x-y\|\, \|(\widehat{x-y})(J(\widehat{x_n-x}), \widehat{x-y})(\widehat{x_n-x})\|$$

$$\leqslant \frac{2\alpha}{\beta}\|x-y\|.$$

Since J is weak to weak continuous we shall obtain, using weak lower-semicontinuity of the norm that,

$$\|fx-fy\|\, \|\widehat{fx-fy} - (Jw, \widehat{fx-fy})\, w\| \leqslant \frac{2\alpha}{\beta}\|x-y\|.$$

Hence as above $w \neq \theta$, since $\alpha < \frac{1}{2}$ and β can be taken arbitrarily close to 1.

Since f is TC I or TC II, the inequality (1) of Theorem 2.2 implies that collinear points θ, $\hat{x}_n$, and x_n are mapped on collinear points. Therefore $f(\hat{x}_n) = \alpha_n fx_n + (1-\alpha_n) f\theta$, for some α_n's. In other words

$$f\hat{x}_n = \alpha_n t_n x_n + (1-\alpha_n) f\theta$$

$$= \alpha_n t_n \|x_n\| \hat{x}_n + (1-\alpha_n) f\theta.$$

Since f is completely continuous we may assume $f\hat{x}_n \to fw$, hence α_n's cannot be unbounded. Also since $t_n \|x_n\| \to \infty$, we assume $\alpha_n \to 0$, and then

$$fw = y + f\theta, \text{ where } y = \lim_{n\to\infty} \alpha_n t_n \|x_n\| \hat{x}_n.$$

Here $y \neq \theta$ by hypothesis, hence $y = tw$ for some t, thus $\hat{x}_n \to w$ and f has the property (a).

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