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THE PROBLEM OF A HOT PUNCH IN A MICROPOLAR ELASTIC SOLID

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The solution of the problem of a hot punch in a homogeneous isotropic, linear micropolar half space has been obtained when the temperature is prescribed within the contact area. The stress and displacement components are obtained in the form of double integrals and the additional radial displacement, normal stress and load on the surface due to the thermal effects have been calculated.

INTRODUCTION

Crack problems in elastic solids with prescribed temperature field have been investigated by many workers[2,4,6-8]. George and Sneddon[5], Chakravorty[1] studied heated punch problems in classical elasticity.

We have considered in this paper a rigid hot punch problem in linear micropolar half space with temperature being constant or having a parabolic distribution under the punch. The problem has been reduced to the solution of a pair of uncoupled dual integral equations which in turn are reduced to a pair of Fredholm integral equations of the second kind. The thermal displacement, microrotation and stress components, the intensity factors and the pressure acting under the hot punch have been derived. It is shown that on the surface, the displacement component normal to the surface, the microrotation and the couple stress μ_{zr} are the same as for an isothermal punch[3], whereas the normal stress, total load and radial displacement differ by amounts which have simple expressions for the particular temperature distribution considered, for any punch profile.

2. FORMULATION OF THE PROBLEM

The solution of the basic equation of the static thermal problems in a micropolar medium ($z \geqslant 0$) in the axisymmetric case is given by[8]

$$u(r, z) = \int_0^\infty \xi \Big[\Big\{ a(\xi) - \frac{\lambda+3\mu'}{\lambda+2\mu'} \xi^{-1} b(\xi) + \frac{\lambda+\mu'}{\lambda+2\mu'} z\, b(\xi)$$

(*equation continued on p.* 1182)

$$+ \frac{\beta_0 z}{2(\lambda+2\mu')} \theta(\xi) \Big\} e^{-\xi z} - \frac{\gamma}{2\mu'} \eta c(\xi) e^{-\eta z} \Big] J_1(\xi r)\, d\xi \quad \ldots(2.1a)$$

$$\omega(r, z) = \int_0^\infty \xi \Big[\Big\{ a(\xi) + \frac{\lambda+\mu'}{\lambda+2\mu'} z\, b(\xi) + \frac{\beta_0}{2(\lambda+2\mu')} (z - \xi^{-1}) \theta(\xi) \Big\} e^{-\xi z}$$

$$- \frac{\gamma}{2\mu'} \xi\, c(\xi) e^{-\eta z} \Big] J_0(\xi r)\, d\xi \quad \ldots(2.1b)$$

$$\phi(r, z) = \int_0^\infty \xi [\, b(\xi) e^{-\xi z} + c(\xi) e^{-\eta z}] J_0(\xi r)\, d\xi \quad \ldots(2.1c)$$

$$T(r, z) = \int_0^\infty \xi \theta(\xi) e^{-\xi z} J_0(\xi r)\, d\xi \quad \ldots(2.1d)$$

where

$$\mu' = \mu + \frac{k}{2}, \quad \eta = \left(\xi^2 + \frac{2\mu' k}{\gamma(\mu+k)} \right)^{1/2}$$

and $\beta_0 = (3\lambda + 2\mu')\, \alpha_t$, λ, μ, k, γ are elastic constants, α_t the coefficient of linear expansion, and $a(\xi)$, $b(\xi)$, $c(\xi)$, $\theta(\xi)$ unknown functions which are to be determined from the boundary conditions of the problem.

We require the following stress components :

$$\tau_{zr} = (\mu + k)\, u_{,z} + \mu\, \omega_{,z} - k\, \phi \quad \ldots(2.2a)$$

$$\tau_{zz} = 2\mu'\, \omega_{,z} + \lambda \left(u_{,r} + \frac{u}{r} + \omega_{,z} \right) - \beta_0 T \quad \ldots(2.2b)$$

$$\mu_{z\theta} = \gamma \phi_{,z}. \quad \ldots(2.2c)$$

The conditions on the boundary are the following :

I. ELASTIC CONDITIONS :

$$\tau_{zz}^{(r,\,0)} = 0,\ \omega(r, 0) = f(r),\ \phi(r) = g(r),\ 0 \leqslant r < a \quad \ldots(2.3a)$$

$$\tau_{zr}(r, 0) = 0,\ \tau_{zz}(r, 0) = 0,\ \mu_{z\theta}(r, 0) = 0,\ r > a. \quad \ldots(2.3b)$$

We assume here that the punch is smooth and rigid, the vertical displacement (ω) and microrotation (ϕ) being prescribed inside the region of contact ($0 \leqslant r < a$).

II. THERMAL CONDITIONS :

$$T(r, 0) = Q(r), \quad 0 \leqslant r < a \qquad \ldots(2.4a)$$

$$= 0, \qquad r > a. \qquad \ldots(2.4b)$$

Temperature has been prescribed on the surface. With the help of the stress-strain relation (2.2a) and the solution (2.1a-c), the boundary conditions $(2.3a)_1$ and $(2.3b)_1$ may be written as

$$\tau_{zr}(r, 0) = 2\mu' \int_0^\infty \xi \Big[-\xi a(\xi) + b(\xi) + \frac{\gamma}{2\mu'} \xi^2 c(\xi) + \frac{\beta_0 \theta(\xi)}{2(\lambda + 2\mu')} \Big] J_1(\xi r)\, d\xi = 0,$$

so that

$$\xi a(\xi) = b(\xi) + \frac{\gamma}{2\mu'} \xi^2 c(\xi) + \frac{\beta_0}{2(\lambda+2\mu')} \theta(\xi). \qquad \ldots(2.5)$$

Defining the nondimensional quantities

$$\xi = \frac{x}{a}, \eta = \frac{y}{a}, \rho = \frac{r}{a}$$

and using (2.5) we record the following quantities from (2.1a-d) and (2.2b-c)

$$u(\rho, 0) = -\frac{a}{2} \Big[\int_0^\infty \Big\{ k_3 B(x) + k_2 x(y-x) C(x) \Big\} J_1(x\rho)\, dx - \frac{\beta_0}{\lambda + 2\mu'} \int_0^\infty \chi(x) J_1(x\rho)\, dx \Big] \qquad \ldots(2.6)$$

$$\omega(\rho, 0) = a \int_0^\infty B(x) J_0(x\rho)\, dx \qquad \ldots(2.7)$$

$$\phi(\rho, 0) = \int_0^\infty x [B(x) + C(x)] J_1(x\rho)\, dx \qquad \ldots(2.8)$$

(equation continued on p. 1184)

$$\tau_{zz}(\rho, 0) = -\mu'\Big[\int_0^\infty x\left\{k_1 B(x) + k_2 x(x-y) C(x)\right\} J_0(x\rho)\, dx + m\int_0^\infty x\,\chi(x)\, J_0(x\rho)\, dx\Big] \quad ...(2.9)$$

$$\mu_{z\theta}(\rho, 0) = -\frac{\gamma}{a}\int_0^\infty x\,[x B(x) + y C(x)]\, J_1(x\rho)\, dx \quad ...(2.10)$$

$$T(\rho, 0) = \int_0^\infty x\,\chi(x)\, J_0(x\rho)\, dx \quad ...(2.11)$$

where we have taken $b\left(\frac{x}{a}\right) = a^2 B(x)$, $c\left(\frac{x}{a}\right) = a^2 C(x)$, $\theta\left(\frac{x}{a}\right) = a^2\chi(x)$ and $k_1 = (1-\nu')^{-1}$, $k_2 = \frac{\gamma}{a^2\mu'}$, $k_3 = 2-k_1$, $m = \frac{1-\nu'}{1+\nu'}\alpha_t$; ν' being the Poisson ratio of the medium given by $\nu' = \frac{\lambda}{2(\lambda+\mu')}$ and $y = \sqrt{x^2 + M^2}$, $M = \sqrt{\frac{2}{k_2}\frac{k}{\mu+k}}$.

Now from the boundary conditions (2.3a-b) and from (2.7-11) we obtain the relations

$$\int_0^\infty x\,[k_1 B(x) + k_2 x(x-y) C(x)]\, J_0(x\rho)\, dx = -m\, T(\rho, 0),\ \rho > 1 \quad ...(2.12)$$

$$\int_0^\infty x\,[x B(x) + y C(x)]\, J_1(x\rho)\, dx = 0,\quad \rho > 1 \quad ...(2.13)$$

$$\int_0^\infty B(x)\, J_0(x\rho)\, dx = F(\rho),\quad 0 \leqslant \rho < 1 \quad ...(2.14)$$

$$\int_0^\infty x\,[B(x) + C(x)]\, J_1(x\rho)\, dx = G(\rho),\quad 0 \leqslant \rho < 1 \quad ...(2.15)$$

where $F(\rho) = a^{-1} f(a\rho)$ and $G(\rho) = g(a\rho)$ are non dimensional quantities.

We may here proceed to deterime $\chi(x)$. With the help of the conditions (2.4a-b), eqn. (2.11) leads to the equations

$$\int_0^\infty x\,\chi(x)\,J_0(x\rho)\,dx = q(\rho),\ 0 \leqslant \rho < 1$$

$$= 0, \qquad q > 1 \qquad \ldots(2.16)$$

with $q(\rho) = Q(a\rho)$.

Solutions of (2.16) may be obtained simply by inversion as

$$\chi(x) = \int_0^1 \rho\, q(\rho)\, J_0(x\rho)\, d\rho \qquad \ldots(2.17)$$

can now be supposed to be known when $q(\rho)$ is given.

We consider two particular cases for the value of $q(\rho)$ as

$$\text{Case (i)}: \quad q(\rho) = q_0 \qquad \ldots(2.18a)$$

$$\text{Case (ii)}: \quad q(\rho) = q_0(1-\rho^2) \qquad \ldots(2.18b)$$

q_0 being a constant, whence

$$\chi(x) = q_0\, x^{-1}\, J_1(x) \qquad \text{in case (i)} \qquad \ldots(2.19)$$

$$= 2q_0\, x^{-2}\, J_2(x) \qquad \text{in case (ii).} \qquad \ldots(2.20)$$

3. Solution to the Problem

Integrating $\rho'\,\tau_{zz}(\rho', 0)$ w.r. to ρ' from 0 to ρ on using $(2.3b)_2$, we obtain

$$\int_0^\infty \left[k_1 B(x) + k_2\, x(x-y)\, C(x)\right] J_1(x\rho)\, dx = \frac{\Lambda}{\rho} - mR(\rho),\ \rho > 1 \qquad \ldots(3.1)$$

with

$$\Lambda = -\frac{1}{\mu'}\int_0^1 \rho\, \tau_{zz}(\rho, 0)\, d\rho \qquad \ldots(3.2)$$

and

$$R(\rho) = \int_0^\infty \chi(x)\, J_1(x\rho)\, dx. \qquad \ldots(3.3)$$

Multiplying both sides of (2.14) by ρ and integrating with respect to ρ from 0 to ρ

$$\int_0^\infty x^{-1} B(x)\, J_1(x\rho)\, dx = H(\rho), \quad 0 \leqslant \rho < 1 \qquad \ldots(3.4)$$

where

$$H(\rho) = \frac{1}{\rho}\int_0^\rho \rho' F(\rho')\, d\rho'. \qquad \ldots(3.5)$$

Combining (2.15) with (3.4) and (2.13) with (3.1) we have

$$\int_0^\infty x^{-1} K(x)\, J_1(x\rho)\, dx = \psi(\rho), \quad 0 \leqslant \rho < 1 \qquad \ldots(3.6)$$

$$\int_0^\infty K(x)\, J_1(x\rho)\, dx = \frac{\Lambda}{\rho} - mR(\rho), \ \rho > 1 \qquad \ldots(3.7)$$

where

$$K(x) = (k_1 + k_2 x^2) B(x) + k_2 x^2 C(x) \qquad \ldots(3.8)$$

$$\psi(\rho) = k_1 H(\rho) + k_2 G(\rho). \qquad \ldots(3.9)$$

The solution of (3.6) and (3.7) is given by (Sneddon[9], p.88)

$$K(x) = V_1(x) + V_2(x) + V_3(x) \qquad \ldots(3.10)$$

where

$$V_1(x) = \sqrt{\frac{2}{\pi}}\, x\left[\sqrt{x}\, J_{1/2}(x) \int_0^1 \frac{\zeta^2 \psi(\zeta)}{\sqrt{1-\zeta^2}}\, d\zeta\right.$$

(equation continued on p. 1187)

$$+ \int_0^1 \frac{\zeta^2\, d\zeta}{\sqrt{1-\zeta^2}} \int_0^1 (xt)^{3/2} J_{3/2}(xt)\, \psi(\zeta t)\, dt \Big] \qquad \ldots(3.11)$$

$$V_2(x) = \sqrt{\frac{2}{\pi}} x^{3/2} \int_1^\infty t^{3/2} J_{1/2}(ut)\, dt \int_1^\infty \frac{\Lambda}{\zeta t} \frac{d\zeta}{\sqrt{\zeta^2-1}} \qquad \ldots(3.12)$$

$$V_3(x) = -\sqrt{\frac{2}{\pi}} x^{3/2} \int_1^\infty t^{3/2} J_{1/2}(ut)\, dt \int_1^\infty mR(\zeta) \frac{d\zeta}{\sqrt{\zeta^2-1}}. \qquad \ldots(3.13)$$

Second term in $K(x)$ can be shown to be

$$V_2(x) = \Lambda \int_{|x|}^\infty \sin u\ du = \Lambda \lim_{\epsilon \to 0} \int_{|x|}^\infty e^{-\epsilon u} \sin u\, du$$

$$= \Lambda \cos x.$$

The contribution of the temperature field arises out in the third term in $K(x)$ which on simplification becomes

$$V_3(x) = -\frac{2m}{\pi} \int_1^\infty (xt) \sin(xt)\, dt \int_1^\infty \frac{R(\zeta t)}{\sqrt{\zeta^2-1}}\, d\zeta.$$

Substituting (3.3), the integral

$$\int_1^\infty \frac{R(\zeta t)}{\sqrt{\zeta^2-1}}\, d\zeta$$

$$= \int_0^\infty \chi(u)\, du \int_1^\infty \frac{J_1(\zeta ut)}{\sqrt{\zeta^2-1}}\, d\zeta$$

$$= \int_0^\infty \frac{\sin(ut)}{ut} \chi(u)\, du. \qquad \ldots(3.14)$$

Inserting $\chi(u)$ from (2.19) and (2.20) in (3.14) and with the results thus obtained, we get

$$V_3(x) = -\Lambda_0 \cos x$$

where

$$\Lambda_0 = \frac{m}{2} q_0 \quad \text{or} \quad \frac{m}{4} q_0 \qquad \ldots(3.15)$$

according as $q(\rho)$ is given by (2.18a) or (2.18b).

Thus we have

$$K(x) = V_1(x) + \Lambda_1 \cos x \qquad \ldots(3.16)$$

where $V_1(x)$ is given by (3.11) and Λ_1 is given by

$$\Lambda_1 = \Lambda - \Lambda_0. \qquad \ldots(3.17)$$

The solution $V_1(x)$ corresponds to punches having circular cross-sections but arbitrary punch profiles. For example, we consider simple punch profiles : (i) parabolic punch and (ii) flat punch. Further in these examples we consider $G(\rho) \equiv 0$ which corresponds to the absence of microrotation under the punch.

Then,

(i) for the parabolic punch

$$F(\rho) = 1 - \rho^2$$

and $V_1(x)$ is given by

$$V_1^{(1)}(x) = \frac{2k_1}{\pi}\left[\frac{4}{x^3}(\sin x - x \cos x) + \frac{\sin x}{x} - \frac{\cos x}{3}\right] \qquad \ldots(3.18)$$

(ii) for the flat punch

$$F(\rho) = 1$$

and $V_1(x)$ is given by

$$V_1^{(2)}(x) = \frac{2k_1}{\pi} \frac{\sin x - x \cos x}{x}. \qquad \ldots(3.19)$$

We have from (2.12), using (3.8)

$$\int_0^\infty \left[\frac{k_1 y + k_2 x^2 (y-x)}{k_1 + N^2} B(x) - \frac{(y-x) K(x)}{k_1 + N^2}\right] J_0(x\rho)\, dx$$

(equation continued on p. 1189)

$$= - \frac{m}{k_1 + N^2} T(\rho, 0), \qquad \rho > 1 \qquad \ldots(3.20)$$

where

$$N^2 = \frac{k}{\mu + k}.$$

If we set

$$D(x) = \frac{k_1 y + k_2 x^2 (y - x)}{k_1 + N^2} B(x) - \frac{(y - x) K(x)}{k_1 + N^2} \qquad \ldots(3.21)$$

we have from (2.14)

$$\int_0^\infty \frac{1 + k(x)}{x} D(x) J_0(x\rho)\, dx = F_1(\rho), \qquad 0 \leqslant \rho < 1 \qquad \ldots(3.22)$$

and from (3.20)

$$\int_0^\infty D(x) J_0(x\rho)\, dx = - \frac{m}{k_1 + N^2} T(\rho, 0), \qquad \rho > 1 \qquad \ldots(3.23)$$

where we have written,

$$\frac{1 + k(x)}{x} = \frac{k_1 + N^2}{k_1 y + k_2 x^2 (y - x)} \qquad \ldots(3.24)$$

and

$$F_1(\rho) = F(\rho) - \int_0^\infty \frac{(y - x) K(x)}{k_1 + N^2} \frac{1 + k(x)}{x} J_0(x\rho)\, dx. \qquad \ldots(3.25)$$

The function $k(x)$ given by (3.24) can be expressed in the form

$$k(x) = \frac{N^2 x (\sqrt{x^2 + M^2} - x) - k_1 M^2}{k_1 (x^2 + M^2) + k_1 x \sqrt{x^2 + M^2} + 2N^2 x^2} \qquad \ldots(3.26)$$

and is found to satisfy the conditions

(i) $k(0) = -1$ and (ii) $k(x) = O\left(\frac{1}{x^2}\right)$ for large x.

The unknown function $D(x)$ satisfies the dual integral equations (3.22) and (3.23) and can now be found as the solution (see Sneddon[9], p. 110) of (3.22) and (3.23) as

$$D(x) = -\frac{2mx}{(k_1 + N^2)\pi} \int_1^\infty \cos(xt)\, dt \int_t^\infty \frac{\rho\, T(\rho, 0)}{\sqrt{\rho^2 - t^2}}\, d\rho + \frac{2}{\pi} x \int_0^1 h(t) \cos(xt)\, dt \quad \ldots(3.27)$$

where the function $h(t)$ appearing in (3.27) is the solution of the integral equation

$$h(x) + \int_0^1 h(t)\, L(x, t)\, dt = I(x) \quad \ldots(3.28)$$

with the Kernel

$$L(x, t) = \frac{2}{\pi} \int_0^\infty k(u) \cos(xu) \cos(tu)\, du \quad \ldots(3.29)$$

and the free term

$$I(x) = \frac{d}{dx} \int_0^x \frac{u F_1(u)}{\sqrt{x^2 - u^2}}\, du + \frac{mx}{k_1 + N^2} \int_1^\infty u\, T(u, 0)\, du \int_0^\infty k(t) \cos(xt)\, J_0(ut)\, dt. \quad \ldots(3.30)$$

We express $D(x)$ as

$$D(x) = D_0(x) + D_1(x) - \Lambda_1 D_2(x) \quad \ldots(3.31)$$

where

$$D_0(x) = -\frac{2mx}{(k_1 + N^2)\pi} \int_1^\infty \cos(xt)\, dt \int_t^\infty \frac{\rho\, T(\rho, 0)}{\sqrt{\rho^2 - t^2}}\, d\rho \quad \ldots(3.32a)$$

$$D_1(x) = \frac{2x}{\pi} \int_0^1 h_1(t) \cos(xt)\, dt \quad \ldots(3.32b)$$

$$h(t) = h_1(t) - \Lambda_1 h_2(t). \quad \ldots(3.32c)$$

After inserting $F_1(u)$ from (3.25) in (3.30), we also express $I(x)$ in the form

$$I(x) = I_0(x) + I_1(x) - \Lambda_1 I_2(x) \qquad \ldots(3.33)$$

where

$$I_0(x) = \frac{mx}{k_1+N^2} \int_1^\infty u\, T(u, 0)\, du \int_0^\infty k(t) \cos(xt)\, J_0(ut)\, dt \qquad \ldots(3.34a)$$

$$I_1(x) = F(0) + x \int_0^{\pi/2} F'(x \sin\theta)\, d\theta - \int_0^\infty \frac{\sqrt{u^2+M^2}-u}{k_1 y + k_2 u^2 (y-u)} V_1(x) \cos(xu)\, du \qquad \ldots(3.34b)$$

$$I_2(x) = \int_0^\infty \frac{\sqrt{u^2+M^2}-u}{k_1 y + k_2 u^2 (y-u)} \cos(xu) \cos u\, du. \qquad \ldots(3.34c)$$

On using $T(\rho, 0)$ from (2.11) the last integral in (3.32a) becomes

$$\int_t^\infty \frac{\rho\, T(\rho, 0)}{\sqrt{\rho^2 - t^2}}\, d\rho$$

$$= \int_0^\infty x \chi(x)\, dx \int_t^\infty \frac{\rho\, J_0(x\rho)}{\sqrt{\rho^2 - t^2}}\, d\rho$$

$$= \int_0^\infty x \chi(x)\, dx \left[t \int_0^\infty \cosh u\, J_0(t x \cosh u)\, du \right]$$

$$= \int_0^\infty \chi(x) \cos(xt)\, dx.$$

Then from (3.32a), $D_0(x)$ is found to be

$$D_0(x) = -\frac{2mx}{(k_1+N^2)\pi} \int_1^\infty \cos(xt)\, dt \int_0^\infty \chi(u) \cos(ut)\, du. \qquad \ldots(3.35)$$

Using the relation

$$J_0(ut) = \frac{2}{\pi} \int_0^u \frac{\cos(ts)\, ds}{\sqrt{u^2 - s^2}}$$

and interchanging the order of integration in $I_0(x)$ we get

$$I_0(x) = \frac{mx}{k_1 + N^2} \int_1^\infty u\, T(u, 0)\, du \int_0^u \frac{L(x, s)\, ds}{\sqrt{u^2 - s^2}}. \qquad \ldots(3.36)$$

Replacing $\chi(u)$ from (2.19) and (2.20) and using the following

$$\int_0^\infty u^{-1} J_1(u) \cos(ut)\, du = 0, \quad t \geqslant 1$$

$$\int_0^\infty u^{-1} J_2(u) \cos(ut)\, du = 0, \quad t > 1$$

$$\int_0^\infty J_1(x) J_0(xu)\, dx = 0, \quad u > 1$$

$$\int_0^\infty x^{-1} J_2(x) J_0(xu)\, dx = 0, \quad u > 1$$

we have in both the cases (i) and (ii)

$$D_0(x) \equiv 0, \; I_0(x) \equiv 0. \qquad \ldots(3.37)$$

By virtue of (3.37), from (3.27)

$$D(x) = D_1(x) - \Lambda_1 D_2(x) \qquad \ldots(3.38)$$

and from (3.38), using (3.32c), we see that $h_i(t)$, $i = 1, 2$ are the solutions of the Fredholm integral equations of the second kind

$$h_i(x) + \int_0^1 h_i(t)\, L(x, t)\, dt = I_i(x), \quad i = 1, 2. \qquad \ldots(3.39)$$

The integrals $L(x, t)$, $I_1(x)$, $I_2(x)$ can be transformed into definite integrals by deforming the contour of integration from real axis to imaginary axis[8]. Now we have

$$L(x, t) = \frac{M}{\pi}\left(1 + \frac{N'^2}{2}\right) \int_0^1 \frac{\zeta \sqrt{1-\zeta^2}\,(N'^2\,\zeta^2 - 1)}{\{1-\zeta^2\,(1+N'^2)\}^2 + \zeta^2\,(1-\zeta^2)} \Big[e^{-M\zeta|e-t|} + e^{-M\zeta\,(x+t)} \Big] d\zeta \qquad \ldots(3.40)$$

$$I_1^{(1)}(x) = (1-2x^2) - \frac{4N'^2}{M^2} + \frac{2}{\pi}\,\text{p.f.}\int_0^1 \left[\frac{4}{M^2\zeta^2} + \frac{4}{M\zeta} + 1 - \frac{1}{3}M\zeta\right] \times \frac{\sqrt{1-\zeta^2}\; e^{-M\zeta}\cosh(M\zeta x)}{\{1-\zeta^2(1+N'^2)\}^2 + \zeta^2(1-\zeta^2)} \qquad \ldots(3.41a)$$

$$I_1^{(2)}(x) = \frac{2}{\pi}\int_0^1 \frac{\zeta\sqrt{1-\zeta^2}\,(1+M\zeta)\,e^{-M\zeta}\cosh(M\zeta x)}{\{1-\zeta^2(1+N'^2)\}^2 + \zeta^2(1-\zeta^2)}\,d\zeta \qquad \ldots(3.41b)$$

$$I_2(x) = \frac{2M}{k_1}\int_0^1 \frac{\zeta\sqrt{1-\zeta^2}\; e^{-M\zeta}\cosh(M\zeta x)}{\{1-\zeta^2(1+N'^2)\}^2+\zeta^2(1-\zeta^2)}\,d\zeta \qquad \ldots(3.42)$$

where

$$N'^2 = \frac{2N^2}{k_1}.$$

4. Components of Displacement, Microrotation and Stress

In this section we find $u(\rho, 0)$, $\omega(\rho, 0)$, $\phi(\rho, 0)$, $\tau_{zz}(\rho, 0)$ and $\mu_{z\theta}(\rho, 0)$ given by (2.6)-(2.10) interms of the known functions $V_1(x)$, $h_1(x)$ and $\chi(x)$.

However, here we consider the case of a flat punch profile so that $V_1(x)$ is given by $V_1^{(2)}(x)$, in (3.19).

Eliminating $B(x)$ and $C(x)$ from (2.6), (2.7) and (2.8) with the help of (3.8) and (3.21) and using the relations (3.16), (3.27) and $(3.37)_1$, we can show that

$$a^{-1}u(\rho, 0) = -\frac{2}{\pi}(k_1+N^2)\int_0^1 h(t)\,dt\int_0^\infty \psi_1(x)\cos(xt)\,J_1(x\rho)\,dx + \frac{k_1+N^2}{\pi}\int_0^1 h(t)\,dt\int_0^\infty J_1(x\rho)\cos(xt)\,dt - \int_0^\infty \psi_2(x)\,[V_1(x) + \Lambda_1\cos x]\,J_1(x\rho)\,dx$$

(*equation continued on p.* 1194)

$$+\frac{\beta_0}{2(\lambda+2\mu')}\int_0^\infty \chi(x)\, J_1(x\rho)\, dx \qquad \ldots(4.1)$$

$$a^{-1}\,\omega(\rho, 0) = \frac{2}{\pi}(k_1 + N^2)\int_0^1 h(t)\, dt \int_0^\infty \psi_1(x)\cos(xt)\, J_0(x\rho)\, dx$$

$$+\int_0^\infty \psi_2(x)\{V_1(x) + \Lambda_1 \cos x\}\, J_0(x\rho)\, dx \qquad \ldots(4.2)$$

$$k_2\,\phi(\rho, 0) = \int_0^\infty \frac{V_1(x) + \Lambda_1 \cos x}{x}\, J_1(x\rho)\, dx$$

$$-k_1\int_0^\infty x^{-1}\,\psi_2(x)\{V_1(x) + \Lambda_1 \cos x\}\, J_1(x\rho)\, dx$$

$$-\frac{2}{\pi}(k_1 + N^2)\, k_1 \int_0^1 h(t)\, dt \int_0^\infty x^{-1}\,\psi_1(x)\, J_1(x\rho)\, dx \qquad \ldots(4.3)$$

$$\tau_{zz}(\rho,0) = -\frac{2\mu'}{\pi}(k_1+N^2)\left[\frac{h(1)}{\sqrt{1-\rho^2}} - \int_\rho^1 \frac{h'(t)\, dt}{\sqrt{t^2-\rho^2}}\right] - \mu' m\, q(\rho),$$

$$0 < \rho < 1 \qquad \ldots(4.4a)$$

$$= 0, \quad \rho > 1 \qquad \ldots(4.4b)$$

$$\mu_{z\theta}(\rho, 0) = \frac{\gamma}{k_2 a}\left[(k_1+N^2)\frac{2}{\pi}\left\{\frac{1}{\rho}\int_0^1 h(t)\, dt - \frac{1}{\rho}\int_\rho^1 \frac{t\, h(t)\, dt}{\sqrt{t^2-\rho^2}}\right\} - S(\rho)\right]$$

$$0 < \rho < 1 \qquad \ldots(4.5a)$$

$$= 0, \quad \rho > 1 \qquad \ldots(4.5b)$$

where

$$\psi_1(x) = \frac{x}{k_1 y + k_2 x^2 (y-x)} \qquad \ldots(4.6a)$$

$$\psi_2(x) = \frac{y-x}{k_1 y + k_2 x^2 (y-x)} \qquad \ldots(4.6b)$$

$$S(\rho) = \int_0^\infty \left\{ V_1^{(2)}(x) + \Lambda_1 \cos x \right\} J_1(x\rho)\, dx$$

$$= \frac{2k_1}{\pi} \frac{\rho}{1+\sqrt{1-\rho^2}} \left[1 + \left(1 - \frac{\pi}{2k_1} \Lambda_1 \right) \frac{1}{\sqrt{1-\rho^2}} \right]. \qquad \ldots(4.7)$$

It is easy to verify that $\omega(\rho, 0)$ and $\phi(\rho, 0)$ agree with the boundary conditions in $0 < \rho < 1$.

Using the formulas

$$J_0(x\rho) = \frac{2}{\pi} \int_0^\rho \frac{\cos(xs)\, ds}{\sqrt{\rho^2 - s^2}}$$

$$J_1(x\rho) = \frac{2}{\pi\rho} \int_0^\rho \frac{s \sin(xs)}{\sqrt{\rho^2 - s^2}}\, ds$$

and the integrals given in Appendices I and II, we get from (4.1) to (4.3)

$$a^{-1} u(\rho, 0) = u_1(\rho) + u_2(\rho) + \frac{k_1 + N^2}{\pi} \left[\frac{1}{\rho} \int_0^\rho h(t)\, dt - \frac{1}{\rho} \int_\rho^1 \frac{t\, h(t)\, dt}{\sqrt{t^2 - \rho^2}} \right]$$

$$+ \frac{\beta_0 q_0}{2(\lambda + 2\mu)} \cdot \frac{\rho}{2}, \quad 0 < \rho < 1 \qquad \ldots(4.8a)$$

$$= u_1(\rho) + u_2(\rho) + \frac{k_1 + N^2}{\pi} \frac{1}{\rho} \int_0^\rho h(t)\, dt$$

$$+ \frac{\beta_0}{2(\lambda^2 + 2\mu')} \cdot \frac{1}{2\rho}, \quad \rho > 1 \qquad \ldots(4.8b)$$

where

$$u_1(\rho) = -\frac{2M}{\pi^2} \left(1 + \frac{N'^2}{2} \right) \int_0^1 h(t)\, dt \int_0^\pi \sin\theta\, d\theta \left[\int_0^1 f_3(\tau)\, G_1(\rho \sin\theta, t, \tau)\, d\tau \right.$$

$$\left. + \int_1^\infty f_4(\tau)\, G_1(\rho \sin\theta, t, \tau)\, d\tau \right]$$

$$u_2(\rho) = \frac{2}{\pi^2}\int_0^{\pi/2} \sin\theta\, d\theta \Big[\Big\{\text{p. f.}\int_0^1 \tau^{-1} f_1(\tau)\, G_2(\rho\sin\theta, t, \tau)\, d\tau + \int_1^\infty \tau^{-1} f_2(\tau)\, G_2(\rho\sin\theta, t, \tau)\, d\tau\Big\} + \Big(1-\frac{\pi}{2k_1}\Lambda_1\Big) M\Big\{\int_0^1 f_1(\tau)\, G_1(\rho\sin\theta, 1, \tau)\, d\tau + \int_1^\infty f_2(\tau)\, G_1(\rho\sin\theta, 1, \tau)\, d\tau\Big\}\Big]$$

$$a^{-1}\omega(\rho, 0) = \frac{2}{\pi^2}\Big(1+\frac{N'^2}{2}\Big) M\int_0^1 h(t)\, dt \int_0^{\pi/2} d\theta\Big[\int_0^1 (N'^2\tau^2-1)\, g(\tau)\, G_3(\rho\sin\theta, t, \tau)\, d\tau\Big] + \frac{2}{\pi^2}\int_0^{\pi/2} d\theta\Big[\int_0^1 \tau^{-1} g(\tau)\{2-G_3(\rho\sin\theta, 1, \tau)\}\, d\tau - \Big(1-\frac{\pi}{2k_1}\Lambda_1\Big) M\int_0^1 g(\tau)\, G_3(\rho\sin\theta, 1, \tau)\, d\tau\Big],\ \rho > 1 \qquad \ldots(4.9)$$

$$k_2\phi(\rho, 0) = -\frac{2k_1}{\pi^2}\int_0^{\pi/2} \sin\theta\, d\theta\Big[\frac{1}{M}\,\text{p. f.}\int_0^1 \tau^{-2} g(\tau)\, G_2(\rho\sin\theta, 1, \tau)\, d\tau - \Big(1-\frac{\pi}{2k_1}\Lambda_1\Big)\int_0^1 \tau^{-1} g(\tau)\{2-G_1(\rho\sin\theta, 1, \tau)\}\, d\tau\Big] - \frac{2}{\pi^2}(k_1+N^2)\int_0^1 h(t)\, dt\int_0^{\pi/2} \sin\theta\, d\theta\Big[\int_0^1 \tau^{-1} f_3(\tau)\, G_3(\rho\sin\theta, t, \tau)\, d\tau + \int_1^\infty \tau^{-1} f_4(\tau)\, G_3(\rho\sin\theta, t, \tau)\, d\tau\Big]$$

(equation continued on p. 1197)

$$+\frac{k_1}{\pi}\left[\rho \sin^{-1}\left(\frac{1}{\rho}\right)-\left(1-\frac{\pi}{2k_1}\Lambda_1\right)\frac{\sqrt{\rho^2-1}}{\rho}\right],\ \rho>1 \qquad \ldots(4.10)$$

where

$$\left.\begin{aligned}
f_1(\tau) &= \frac{1-(1+N'^2)\,\tau^2}{r(\tau)}\\
f_2(\tau) &= -\frac{\sqrt{\tau^2-1}\,(\sqrt{\tau^2-1}\ -\tau)+N'^2\,\tau}{r(\tau)}\\
&= -\frac{1}{\tau\sqrt{\tau^2-1}+(1+N'^2)\,\tau^2-1}\\
f_3(\tau) &= \frac{N'^2\,\tau^4}{r(\tau)}\\
f_4(\tau) &= \frac{\tau\sqrt{\tau^2-1}\,(N'^2\,\tau^2-1)+N'^2\,\tau^4}{r(\tau)}\\
&= \frac{\tau\,(\tau+\sqrt{\tau^2-1})}{\tau\sqrt{\tau^2-1}+(1+N'^2)\,\tau^2-1}\\
g(\tau) &= \frac{\tau\sqrt{1-\tau^2}}{r(\tau)}\\
r(\tau) &= 1-\tau^2(1+2N'^2)+\tau^4(N'^4+2N'^2)
\end{aligned}\right\} \qquad \ldots(4.11)$$

$$\left.\begin{aligned}
G_1(s,t,\tau) &= e^{-M(s+t)\tau}+\operatorname{sgn}(s-t)\,e^{-M|s-t|\tau}\\
G_2(s,t,\tau) &= e^{-M(s+t)\tau}-e^{-M|s-t|\tau}\\
G_3(s,t,\tau) &= e^{-M(s+t)\tau}+e^{-M|s-t|\tau}.
\end{aligned}\right\} \qquad \ldots(4.12)$$

The value of Λ_1 can be obtained from (3.17) and (3.2) with the help of (2.9) as

$$\Lambda_1=\int_0^\infty\left\{k_1\,B(x)+k_2x\,(x-y)\,C(x)\right\}J_1(x)+m\int_0^\infty\chi(x)\,J_1(x)\,dx-\Lambda_0$$

which on using (3.8) and (3.21) becomes

$$\Lambda_1=(k_1+N^2)\int_0^\infty\frac{D(x)}{x}\,J_1(x)\,dx+m\int_0^\infty\chi(x)\,J_1(x)\,dx-\Lambda_0.$$

Since

$$\int_0^\infty x^{-1}\,J_1^2(x)\,dx=\frac{1}{2}$$

and

$$\int_0^\infty x^{-2} J_1(x) J_2(x)\, dx = \frac{1}{8}.$$

$D(x)$ and $\chi(x)$ are respectively given by (3.38) and (2.19), it can be shown that

$$\Lambda_1 = (k_1 + N^2) \int_0^\infty \left[D_1(x) - \Lambda_1 D_2(x) \right] \frac{J_1(x)}{x}\, dx. \qquad \ldots(4.13)$$

Replacing $D_i(x)$, $i = 1, 2$ from (3.32b) in (4.13) and simplifying, we get

$$\Lambda_1 = \frac{\int_0^1 h_1(t)\, dt}{\frac{\pi}{2(k_1 + N^2)} + \int_0^1 h_2(t)\, dt}. \qquad \ldots(4.14)$$

The state of stress near the ends of the punch profile depends on stress intensity factors. The stress intensity factor (SIF) and the couple stress intensity factor (CIF) are defined as

$$p_1 = \left| \lim_{\rho \to 1-} \sqrt{2(1-\rho)}\ \tau_{zz}(\rho, 0) \right| \qquad \ldots(4.15)$$

$$p_2 = \left| \lim_{\rho \to 1-} \sqrt{2(1-\rho)}\ \mu_{z\theta}(\rho, 0) \right|. \qquad \ldots(4.16)$$

By virtue of (4.4) and (4.5) we derive from (4.15) and (4.16) respectively

$$p_1 = \frac{2\mu'}{\pi} (k_1 + N^2)\, h(1) \qquad \ldots(4.17)$$

$$p_2 = \frac{2\gamma}{\pi a} \frac{k_1}{k_2} \left[\frac{\pi}{2k_1} \Lambda_1 - 1 \right]. \qquad \ldots(4.18)$$

The total force exerted by the punch in given by

$$P = -2\pi a^2 \int_0^1 \rho\, \tau_{zz}(\rho, 0)\, d\rho.$$

Using (3.2) and (3.17) we have

$$P = 2\pi \mu' a^2 (\Lambda_1 + \Lambda_0) \qquad \ldots(4.19)$$

where Λ_1, Λ_0 are given by (4.14) and (3.15) respectively.

5. Conclusion

The integral equations (3.39) are identical with those for the isothermal punch (see De Sarkar and Mitra[3]) and Λ_1 is identical with the corresponding constant Λ taken for the isothermal punch.

For a comparison between the hot punch problem and isothermal punch problem for any punch profile we state the following :

$$[\omega (\rho, 0), \psi (\rho, 0), \mu_{z\theta} (\rho, 0)]_H = [\omega (\rho, 0), \phi (\rho, 0), \mu_{z\theta} (\rho, 0)]_I \qquad \ldots(5.1)$$

$$[a^{-1} u (\rho, 0)]_H - [a^{-1} u (\rho, 0)]_I = \frac{\beta_0 q_0}{2(\lambda+2\mu')} \cdot \frac{1}{2} \left[\rho H(1-\rho) + \frac{1}{\rho} H(\rho-1) \right] \qquad \ldots(5.2a)$$

in case (i),

$$= \frac{\beta_0 q_0}{2(\lambda + 2\mu')} \cdot \frac{1}{4} \left[\rho (2 - \rho^2) H(1 - \rho) + \frac{1}{\rho} H(\rho - 1) \right] \qquad \ldots(5.2b)$$

in case (ii).

$$[\tau_{zz} (\rho, 0)]_H - [\tau_{zz} (\rho, 0)]_I = -\mu' m q (\rho), \quad 0 < \rho < 1$$
$$= 0, \quad \rho > 1 \qquad \ldots(5.3)$$

$$[p_1]_H = [p_1]_I, \quad [p_2]_H = [p_1]_I \qquad \ldots(5.4)$$

$$[P]_H - [P]_I = 2\pi \mu' a^2 \Lambda_0 \qquad \ldots(5.5)$$

where Λ_0 is given by (3.15). The suffixes H and I respectively refer to the corresponding quantities in case of the hot punch and isothermal punch problem.

From (5.2a–b), it is observed that due to the temperature distribution (2.18a) and (2.18b) there is a maximum additional radial displacement at the edge of the punch amounting to $\frac{a \beta_0 q_0}{4(\lambda+2\mu')}$ and $\frac{a \beta_0 q_0}{8(\lambda+2\mu')}$ respectively and also from (5.5) it is found that and extra load $2\pi \mu' a^2 \Lambda_0$ is needed to produce the same displacement under the punch.

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APPENDIX I

The following integrals have been evaluated by deforming the contour of integration from real axis to imaginary axis[8].

(i) $$\int_0^\infty \frac{x \sin \zeta x}{k_1 y + k_2 x^2 (y-x)}\, dx = \frac{M}{k_1}\left\{\int_0^1 f_3(\tau)\, e^{-M\zeta r}\, d\tau + \int_1^\infty f_4(\tau)\, e^{-M\zeta r}\, d\tau\right\}$$

(ii) $$\int_0^\infty \frac{(y-x)\cos \zeta x}{x\{k_1 y + k_2 x^2 (y-x)\}}\, dx = \frac{1}{k_1}\left\{\text{p. f.} \int_0^1 \tau^{-1} f_1(\tau)\, e^{-M\zeta r}\, d\tau + \int_1^\infty \tau^{-1} f_2(\tau)\, e^{-M\zeta r}\, d\tau\right\}$$

(iii) $$\int_0^\infty \frac{(y-x)\sin \zeta x}{k_1 y + k_2 x^2 (y-x)}\, dx = \frac{M}{k_1}\left\{\int_0^1 f_1(\tau)\, e^{M\zeta r}\, d\tau + \int_1^\infty f_2(\tau)\, e^{-M\zeta r}\, d\tau\right\}$$

(iv) $$\int_0^\infty \frac{x \cos \zeta x}{k_1 y + k_2 x^2 (y-x)}\, dx = \frac{M}{k_1}\int_0^1 (N'^2 \tau^2 - 1)\, g(\tau)\, e^{-M\zeta r}\, d\tau$$

(v) $$\int_0^\infty \frac{(y-x)\sin \zeta x}{x\{k_1 y + k_2 x^2 (y-x)\}}\, dx = \frac{1}{k_1}\int_0^1 \tau^{-1} g(\tau)\{1 - e^{-M\zeta r}\}\, d\tau$$

(vi) $$\int_0^\infty \frac{(y-x)\cos \zeta x}{k_1 y + k_2 x^2 (y-x)}\, dx = \frac{M}{k_1}\int_1^1 g(\tau)\, e^{-M\zeta r}\, d\tau$$

(vii) $$\int_0^\infty \frac{\cos \zeta x}{k_1 y + k_2 x^2 (y-x)}\, dx = \frac{1}{k_1}\left\{ \int_0^1 \tau^{-1} f_3(\tau)\, e^{-M\zeta\tau}\, d\tau + \int_1^\infty \tau^{-1} f_4(\tau)\, e^{-M\zeta\tau}\, d\tau \right\}$$

(viii) $$\int_0^\infty \frac{(y-x)\cos \zeta x}{x^2 \{k_1 y + k_2 x^2 (y-x)\}}\, dx = -\frac{1}{Mk_1}\, \text{p. f.} \int_0^1 \tau^{-2} g(\tau)\, e^{-M\zeta\tau}\, d\tau.$$

Appendix II

(i) $$\int_0^\infty \frac{\sin x}{x^2} J_1(x\rho)\, dx = \frac{\sqrt{\rho^2 - 1}}{2\rho} + \frac{\rho}{2} \sin^{-1}\left(\frac{1}{\rho}\right), \quad \rho > 1.$$