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GENERALIZATIONS OF FIXED POINT THEOREMS OF MEIR AND KEELER TYPE

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In section I, two fixed point theorems for three self maps are established. Our first theorem generalises the result of Meir and Keeler[2]. It is analogous to the results of Chung[1] and Park and Bae[4]. Our second theorem generalizes the results of other workers[1-5]. It also improves Theorem 2 of Park and Rhoades[5]. Examples are given to justify our claims.

In section II, a fixed point theorem for a multi valued map which commutes with a single valued map and satisfies a Meir and Keeler type condition is established. It generalizes the above results.

SECTION I

Meir and Keeler[2] proved the following :

Theorem 1.1—Let (X, d) be a complete metric space, $f : X \to X$ satisfies the following condition :

Giving $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leqslant d(x, y) < \epsilon + \delta \text{ implies } d(fx, fy) < \epsilon.$$

Then f has a unique fixed point $z \in X$ and the sequence $\{f^n x\}$ converges to z for all $x \in X$.

Various extensions of the above result have been obtained, in particular, Chung[1] (Th. 1), Pal and Maiti[3] (Th. 2), Park and Bae[4] (Th. 2.4) and Park and Rhoades[5] (Th. 2, Th. 4).

We first give the following definitions :

Let P, Q and T be self maps on a metric space (X, d). If for a point $x_0 \in X$, there exists a sequence $\{x_n\}$ such that $Tx_{2n+1} = Px_{2n}$, $Tx_{2n+2} = Qx_{2n+1}$, $n = 0, 1, 2, \ldots$, then $\mathcal{O}(P, Q, T, x_0) = \{Tx_n \mid n = 1, 2, \ldots\}$ is called the orbit for (P, Q, T) at x_0. X is

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called (P, Q, T) orbitally complete at x_0 iff every Cauchy sequence in $\mathcal{G}(P, Q, T, x_0)$ converges in X. X is called (P, Q, T) orbitally complete if it is so that at every $x \in X$. P, Q and T are said to be orbitally continuous at x_0 iff they are continuous on $\mathcal{G}(P,Q,T,x_0)$.

We now prove the following result.

Theorem 1.2—Let (X, d) be a metric space, $P, Q, T : X \to X$ be such that either $PT = TP$ or $QT = TQ$ and $P(X) \cup Q(X) \subseteq T(X)$; X be (P, Q, T) orbitally complete and also satisfies the following :

Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leqslant \max \{d(Tx, Ty), \tfrac{1}{2}[d(Tx, Px) + d(Ty, Qy)], \tfrac{1}{2}[d(Tx, Qy) + d(Ty, Px)]\} < \epsilon + \delta$$

implies

$$d(Px, Qy) < \epsilon \qquad \ldots(\mathrm{I})$$

further one of the following holds :

(i) $d(Qx, Ty) \leqslant \operatorname{Max}\{d(y, Qx), d(y, Tx)\}$ for all x, y with R. H. S. positive.

(ii) $d(Px, Ty) \leqslant \operatorname{Max}\{d(y, Tx), d(y, Px)\}$ for all x, y with R. H. S. positive.

Then P, Q and T have a unique common fixed point $z \in X$ and for any $x_0 \in X$, every sequence $\{Tx_n\}$ defined by $Tx_{2n+1} = Px_{2n}$, $Tx_{2n+2} = Qx_{2n+1}$, $n = 0, 1, 2, \ldots$ converges to z.

Proof : Without loss of generality we can assume that $PT = TP$. Clearly from (I), we have

$$d(Px, Qy) < \operatorname{Max}\{d(Tx, Ty), \tfrac{1}{2}[d(Tx, Px) + d(Ty, Qy)], \tfrac{1}{2}[d(Tx, Qy) + d(Ty, Px)]\} \text{ whenever R. H. S.} > 0. \qquad \ldots(\mathrm{II})$$

Let $x_0 \in X$, define $Tx_{2n+1} = Px_{2n}$ and $Tx_{2n+2} = Qx_{2n+1}$, $n = 0, 1, 2, \ldots$. This selection is possible since $P(X) \cup Q(X) \subseteq T(X)$.

Case (i)—Suppose $Tx_n = Tx_{n+1}$ for some integer n. We first assume n is odd, If $Tx_{n+1} \neq Tx_{n+2}$ then by (II)

$$d(Tx_{n+2}, Tx_{n+1}) = d(Px_{n+1}, Qx_n)$$
$$< \operatorname{Max}\{d(Tx_{n+1}, Tx_n), \tfrac{1}{2}[d(Tx_{n+1}, Px_{n+1}) + d(Tx_n, Qx_n)], \tfrac{1}{2}[d(Tx_{n+1}, Qx_n) + d(Tx_n, Px_{n+1})]\}$$

i.e., $d(Tx_{n+2}, Tx_{n+1}) < d(Tx_{n+2}, Tx_{n+1})$, a contradiction. Hence $Tx_{n+2} = Tx_{n+1}$. By proceeding in this way, we have $Tx_{n+i} = Tx_n$, $i = 1, 2, \ldots$

Since $Tx_n = Tx_{n+1}$ we have $Tw = Qw$ where $w = x_n$.

Suppose $Pw \neq Qw$. Then by (II) we have

$$d(Pw, Qw) < \tfrac{1}{2}[d(Tw, Pw) + d(Tw, Qw)]$$

so that $d(Pw, Qw) < \frac{1}{2} d(Pw, Qw)$, a contradiction. Hence $Pw = Qw = Tw$.

Since $PT = TP$, we havt $PTw = TPw = TTw$.

Suppose $TTw \neq Tw$. Then by (II), we have

$$d(PTw, Qw) < \text{Max}\,\{d(TTw, Tw), \tfrac{1}{2}[d(TTw, PTw) + d(Tw, Qw)], \tfrac{1}{2}[d(TTw, Qw) + d(Tw, PTw)]\}$$

so that $d(TTw, Tw) < d(TTw, Tw)$, a contradiction. Hence $Tw = TTw = PTw$. Also $QTw = Tw$; for otherwise, we get a contradiction by the application of (II) to $d(Pw, QTw)$.

Thus Tw is a common fixed point of P, Q and T. Uniqueness follows easily. Clearly $\{Tx_n\}$ converges to Tw.

Similar is the case when n is even. Thus the theorem is proved.

Case (ii)—$Tx_n \neq Tx_{n+1}$ for all integers n.

Clearly from (II), $\{d(Tx_n, Tx_{n+1})\}$ is a decreasing sequence of positive terms and hence converges, say, to $r \geqslant 0$. Suppose $r > 0$.

Then there exists $s > 0$ such that

$$r \leqslant \text{Max}\,\{d(Tx, Ty), \tfrac{1}{2}[d(Tx, Px) + d(Ty, Qy)], \tfrac{1}{2}[d(Tx, Qy) + d(Ty, Px)]\} < r + s$$

implies ...(III)

$$d(Px, Qy) < r.$$

Since $\{d(Tx_n, Tx_{n+1})\} \downarrow r$, there exists a positive integer N such that $r \leqslant d(Tx_n, Tx_{n+1}) < r+s$ for all $n > N$. Let $2k \geqslant N$. Noting that,

$$\text{Max}\,\{d(Tx_{2k}, Tx_{2k+1}), \tfrac{1}{2}(d(Tx_{2k}, Px_{2k}) + d(Tx_{2k+1}, Qx_{2k+1})], \tfrac{1}{2}[d(Tx_{2k}, Qx_{2k+1}) + d(Tx_{2k+1}, Px_{2k})]\} = d(Tx_{2k}, Tx_{2k+1})$$

by (III), we get that $d(Px_{2k}, Qx_{2k+1}) < r$, i.e, $d(Tx_{2k+1}, Tx_{2k+2}) < r$ which is a contradiction. Hence $r = 0$. Thus $\{d\, Tx_n, (Tx_{n+1})\} \downarrow 0$.

We now claim that $\{Tx_n\}$ is a Cauchy sequence.

Suppose not. Then there exists an $\epsilon > 0$ such that for each positive integer N there exist integers m, n with $m > n > N$ such that

$$d(Tx_m, Tx_n) \geqslant 2\epsilon.$$

Choose a number δ, $0 < \delta < \epsilon$ for which (I) is satisfied.

Since $\{ d(Tx_n, Tx_{n+1})\} \downarrow 0$, there exists an integer $N = N(\delta)$ such that

$$d(Tx_i, Tx_{i+1}) < \frac{\delta}{6} \text{ for all } i \geqslant N.$$

With this choice of N, pick m, n with $m > n > N$ such that

$$d(Tx_m, Tx_n) \geqslant 2\epsilon > \epsilon + \delta. \qquad \text{...(IV)}$$

From (IV), it is clear that $m-n > 6$, otherwise,

$$d(Tx_m, Tx_n) \leqslant \sum_{i=0}^{5} d(Tx_{n+i}, Tx_{n+i+1}) < \delta < \epsilon + \delta$$

contradicting (IV). Without loss of generality, we may assume that n is even. Suppose

$$d(Tx_n, Tx_{m-1}) < \epsilon + \frac{\delta}{3}$$

then

$$d(Tx_n, Tx_m) \leqslant d(Tx_n, Tx_{m-1}) + d(Tx_{m-1}, Tx_m) < \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \delta$$

which is a contradiction to (IV). So

$$d(Tx_n, Tx_{m-1}) \geqslant \epsilon + \frac{\delta}{3}.$$

Similarly, suppose

$$d(Tx_n, Tx_{m-2}) < \epsilon + \frac{\delta}{3}$$

then

$$d(Tx_n, Tx_m) \leqslant d(Tx_n, Tx_{m-2}) + d(Tx_{m-2}, Tx_{m-1}) + d(Tx_{m-1}, Tx_m)$$

$$< \epsilon + \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} < \epsilon + \delta$$

which is a contradiction to (IV). So

$$d(Tx_n, Tx_{m-2}) \geqslant \epsilon + \frac{\delta}{3}.$$

Thus there exists an odd integer $j > n$ such that $d(Tx_n, Tx_j) \geqslant \epsilon + \frac{\delta}{3}$. Hence there exists a smallest odd integer $j > n$ such that

$$d(Tx_n, Tx_j) \geqslant \epsilon + \frac{\delta}{3}.$$

Now,

$$d(Tx_n, Tx_j) \leqslant d(Tx_n, Tx_{j-2}) + d(Tx_{j-2}, Tx_{j-1})$$

$$+ d(Tx_{j-1}, Tx_j) < \epsilon + \frac{2}{3}\delta.$$

Thus there exists an odd integer $j \in (n, m)$ such that

$$\epsilon + \frac{\delta}{3} \leqslant d(Tx_n, Tx_j) < \epsilon + \frac{2}{3}\delta. \qquad \ldots(V)$$

Since,

$$\epsilon < \epsilon + \frac{\delta}{3} \leqslant d(Tx_n, Tx_j) \leqslant \text{Max}\,\{d(Tx_n, Tx_j), \frac{1}{2}[d(Tx_n, Px_n)$$

$$+ d(Tx_j, Qx_j)], \frac{1}{2}[d(Tx_n, Qx_j) + d(Tx_j, Px_n)]\}$$

$$\leqslant \frac{1}{2}[d(Tx_n, Tx_j) + d(Tx_j, Tx_{j+1}) + d(Tx_j, Tx_n)$$

$$+ d(Tx_n, Tx_{n+1})] < \epsilon + \frac{2}{3}\delta + \frac{\delta}{6} < \epsilon + \delta$$

by (I), $d(Px_n, Qx_j) < \epsilon$, i.e., $d(Tx_{n+1}, Tx_{j+1}) < \epsilon$.

But

$$d(Tx_n, Tx_j) \leqslant d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{j+1}) + d(Tx_{j+1}, Tx_j)$$

(equation continued on p. 1254)

$$< \frac{\delta}{6} + \epsilon + \frac{\delta}{6} = \epsilon + \frac{\delta}{3} \text{ which contradicts (V).}$$

Hence $\{Tx_n\}$ is a Cauchy sequence. Since X is (P, Q, T) orbitally complete there exists a $z \in X$ such that $Tx_n \to z$ as $n \to \infty$.

Clearly, $Tx_n \neq z$ for infinitely many n. We can as well assume that $Tx_n \neq z$ for all n.

If (i) holds, then $d(Qx_{2n+1}, Tz) \leqslant \text{Max}\{d(z, Qx_{2n+1}), d(z, Tx_{2n+1})\}$. Letting $n \to \infty$, we get that $d(z, Tz) \leqslant 0$ which implies that $Tz = z$. If (ii) holds then also $Tz = z$.

Since $Tx_{2n}, \neq Tx_{2n+1}$ by (II) we have

$$d(Px_{2n}, Qz) < \text{Max}\{d(Tx_{2n}, Tz), \tfrac{1}{2}[d(Tx_{2n}, Px_{2n}) + d(Tz, Qz)]$$
$$\tfrac{1}{2}[d(Tx_{2n}, Qz) + d(Tz, Px_{2n})]\}.$$

Letting $n \to \infty$, we get that

$$d(z, Qz) \leqslant \text{Max}\{d(z, Tz), \tfrac{1}{2}[d(z, z) + d(Tz, Qz)], \tfrac{1}{2}[d(z, Qz) + d(Tz, z)]\}$$

i.e. $d(z, Qz) \leqslant \frac{1}{2} d(z, Qz)$ which implies that $Qz = z$.

Applying (II) to $d(Pz, Qz)$ we get $Pz = z$. Thus $Pz = Qz = Tz = z$.

Remarks

(1) By putting $P = Q = f, T = I$ (Identity map) in Theorem 1.2, we get a result which generalizes the Theorem of Meir and Keeler[2] (withour any additional hypothesis since conditions (i) and (ii) of Theorem 1.2 are automatically satisfied).

(2) By putting $P = Q = f^2, T = f$ in Theorem 1.2, we get an independent result similar to Theorem 1 of Chung[1] (see Example 1.4 below).

(3) By putting $P = Q = g, T = f$ in Theorem 1.2, we get an independent result similar to Theorem 2.4 of Park and Bae[4] (see Example 1.5 below).

(4) In the condition (I) of Theorem 1.2, $\frac{1}{2}[d(Tx, Px) + d(Ty, Qy)]$ can not be replaced by Max $\{d(Tx, Px), d(Ty, Qy)\}$ (see example 1.7, below).

(5) In Theorem 1.2, $PT = TP$ or $QT = TQ$ can not be dropped in view of example 1.8, below.

However, we have :

Theorem 1.3—Let P, Q and T be self maps on a metric space (X, d) such that $P(X) \cup Q(X) \subseteq T(X)$, X be (P, Q, T) orbitally complete and satisfying the following condition :

Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leqslant \text{Max}\,\{ d(Tx, Ty), d(Tx, Px), d(Ty, Qy), \tfrac{1}{2}[d(Tx, Qy) + d(Ty, Px)]\} < \epsilon + \delta$$

implies

$$d(Px, Qy) < \epsilon. \qquad \ldots(\text{I}')$$

Further, let $PT = TP$, P, T be orbitally continuous or $QT = TQ$, Q, T be orbitally continuous on X. Then P, Q and T have a unique common fixed point, say, $p \in X$ and for any $x_0 \in X$, the sequence $\{Tx_n\}$ defined as in Theorem 1.2 converges to $z \in X$ satisfying $Tz = p$.

PROOF : Without loss of generality assume that $PT = TP$, P, T are orbitally continuous. Clearly by (I′), for R. H. S. > 0 we have

$$d(Px, Qy) < \text{Max}\,\{d(Tx, Ty), d(Tx, Px), d(Ty, Qy), \tfrac{1}{2}[d(Tx, Qy) + d(Ty, Px)]\}. \qquad \ldots(\text{II}')$$

If $Tx_n = Tx_{n+1}$ for some integer n then as in Case (i) of Theorem 1.2 follows the result. Otherwise, as in Case (ii) of Theorem 1.2, we have $\{Tx_n\}$ converges to $z \in X$.

Then $Pz = \lim_{n\to\infty} PTx_{2n} = \lim_{n\to\infty} TPx_{2n} = Tz.$

Clearly by (II′), follows that $Pz = Qz$. Thus $Pz = Qz = Tz$.

The remaining part of the proof follows as in Case (i) of Theorem 1.2. Theorem 1.3 generalizes the results of the previous works[1-5] and also improves the following :

Theorem (Th. 2, Park and Rhodes[5])—Let f, g be orbitally continuous self maps on a (f, g) orbitally complete metric space (X, d) satisfying the following condition :

Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leqslant \text{Max}\,\{d(x, y), d(x, fx), d(y, gy), \tfrac{1}{2}[d(x, gy) + d(y, fx)]\} < \epsilon + \delta$$

implies

$$d(fx, gy) < \epsilon. \qquad \ldots(\text{I}'')$$

Then for each $x_0 \in X$, either (i) f or g has a fixed point in the (f, g) orbit $\{x_n\}$ of x_0 or (ii) f and g have a common unique fixed point p and $\lim_{n \to \infty} x_n = p$.

Example 1.4—Let $X = \{0\} \cup [1, \infty)$ with usual metric d.

Let $f : X \to X$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Example 1.5—Let $X = [0, 1]$ with the usual metric d. Let $f, g : X \to X$ be defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x \in [0, 1) \\ \frac{1}{4}, & x = 1 \end{cases} \quad \text{and } g(x) = \begin{cases} \frac{x}{4}, & x \in [0, 1) \\ \frac{1}{8}, & x = 1. \end{cases}$$

Here we take $\delta(\epsilon) = \epsilon$.

We enrich our Theorem 1.2 by the following example.

Example 1.6—Let $X = [0, 1]$ with the usual metric d. Let $P, Q, T : X \to X$ be defined by

$$Px = \begin{cases} 0, & x \in [0, 1) \\ \frac{1}{16}, & x = 1 \end{cases}, \quad Q(x) = 0, \ x \in [0, 1]$$

$$Tx = \begin{cases} \frac{x}{2}, & x \in [0, 1) \\ \frac{3}{4}, & x = 1. \end{cases}$$

All the conditions of Theorem 1.2 are satisfied with (i). Here we take $\delta = \epsilon$.

Example 1.7—Let $X = [0, 1]$ with the usual metric. Let $f : X \to X$ be given by

$$fx = \frac{x}{2}, \ x \neq 0, \ f0 = 1.$$

All the conditions of Theorem 1.2 except (I) are satisfied with

$$P = Q = f, \ T = I$$

f satisfies the following conditions :

Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leqslant \text{Max}\,\{d(x, y), d(x, fx), d(y, fy), \tfrac{1}{2}[d(x, fy) + d(y, fx)]\} < \epsilon + \delta$$

implies

$$d(fx, fy) < \epsilon.$$

Here

$$\delta(\epsilon) = \begin{cases} \text{Min}\,\{1 - \epsilon, \epsilon\}, \text{ if } \epsilon \in (0, 1) \\ \epsilon, \text{ if } \epsilon \geqslant 1. \end{cases}$$

But f has no fixed point.

Example 1.8 — Let $X = [0, 1]$ with the usual metric d, $P, Q, T : X \to X$ be defined by $Px = Qx = 0$ for all $x \in X$, and

$$Tx = \begin{cases} 0, x \neq 0 \\ 1, x = 0. \end{cases}$$

SECTION II

We now prove a theorem for a multivalued map that commutes with a single valued map which satisfies Meir and Keeler type condition.

We shall follow the following notations and definitions. Let (X, d) be a metric space. $CL(X) = \{A / A \text{ is a nonempty closed subset of } X\}$.

For

$$A \in CL(X), \epsilon > 0$$

$$N(\epsilon, A) = \{x \in X \mid d(x, a) < \epsilon \text{ for } a \in A\}.$$

For

$$A, B \in CL(X),$$

$$H(A, B) = \begin{cases} \text{Inf}\,\{\epsilon > 0 \mid A \subseteq N(\epsilon, B), B \subseteq N(\epsilon, A)\}, \text{ if exists} \\ \infty, \text{ otherwise.} \end{cases}$$

H is called the generalized Hausdorff distance function for $CL(X)$ induced by d. $D(x, A)$ will denote the distance between x and the set A. Let h be a single valued map from X to itself and T be a multivalued map from X to the non-empty subsets of X.

Definitions—If for a point $x_0 \in X$, there exists a sequence $\{x_n\}$ such that $hx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$, then $O_h(x_0) = \{hx_n \mid n = 1, 2, \ldots\}$ is called the orbit for (T, h) at x_0 Further $O_h(x_0)$ is called a regular orbit for (T, h) if for each pair of non-negative integers i, j

$$d(hx_{i+1}, hx_{j+1}) \leqslant H(Tx_i, Tx_j).$$

A space X is called (T, h, x_0) orbitally complete iff every Cauchy sequence of the form $\{hx_n \mid hx_n \in Tx_{n-1}, n = 1, 2, \ldots\}$ converges in X. A point x is said to be a fixed point of T if $x \in Tx$.

Theorem 2.1—Let (X, d) be a metric space, $T : X \to CL(X)$, $h : X \to X$. Let $A = \bigcup_{a \in X} T(a)$. Let T be continuous on A, h be continuous on $\overline{h(X)}$, H be a metric on A, $hTx \subseteq Thx$ for all $x \in X$ and for given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leqslant \text{Max}\{d(hx, hy), D(hx, Tx), D(hy, Ty), \tfrac{1}{2}[D(hx, Ty) + D(hy, Tx)]\} < \epsilon + \delta$$

implies

$$H(Tx, Ty) < \epsilon. \qquad \ldots(\text{I}')$$

And also assume that for each $x \in X$, there exists a sequence $\{x_n\}$ such that $O_h(x)$ is regular and X is (T, h, x) orbitally complete. Then either there exists a $p \in X$ such that $hp \in Tp$ or for each $x \in X$, the sequence $\{hx_n\}$ with $hx_n \in Tx_{n-1}$, $n = 1, 2, \ldots$ where $x_0 = x$, converges to $z \in X$ with $hz \in Tz$ provided $hx_{n+1} \neq hx_n$ for all n.

PROOF: By (I),

$$H(Tx, Ty) < \text{Max}\{d(hx, hy), D(hx, Tx), D(hy, Ty), \tfrac{1}{2}[D(hx, Ty) + D(hy, Tx)]\} \text{ for } hx \neq hy. \qquad \ldots(\text{II})$$

Let $x \in X$. Since $O_h(x)$ is regular then for each pair of non-negative integers i, j, we have

$$d(hx_{i+1}, hx_{j+1}) \leqslant H(Tx_i, Tx_j) \qquad \ldots(\text{III})$$

Suppose $hx_{n+1} = hx_n$ for some n.

Since $hx_{n+1} \in Tx_n$, we have $hhx_{n+1} \in hTx_n \subseteq Thx_n$. That is $h(hx_n) \in T(hx_n)$.

Thus the theorem is proved.

Assume that $hx_{n+1} \neq hx_n$ for all n.

By (III) and (II),

$$d(hx_{n+1}, hx_{n+2}) \leqslant H(Tx_n, Tx_{n+1})$$

$$< \text{Max}\{d(hx_n, hx_{n+1}), D(hx_n, Tx_n), D(hx_{n+1}, Tx_{n+1}),$$

$$\tfrac{1}{2}[D(hx_n, Tx_{n+1}), D(hx_{n+1}, Tx_n)]\}$$

$$\leqslant \text{Max}\{d(hx_n, hx_{n+1}), d(hx_n, hx_{n+1}), d(hx_{n+1}, hx_{n+2}),$$

$$\tfrac{1}{2}[d(hx_n, hx_{n+2}) + d(hx_{n+1}, hx_{n+1})]\}$$

which implies that,

$$d(hx_{n+1}, hx_{n+2}) \leqslant d(hx_n, hx_{n+1}).$$

Thus $\{d(hx_n, hx_{n+1})\}$ is a decreasing sequence of positive terms and hence converges, say, to $r \geqslant 0$. Suppose $r > 0$.

Then there exists a $s > 0$ such that

$$r \leqslant \text{Max}\{d(hx, hy), D(hx, Tx), D(hy, Ty), \tfrac{1}{2}[D(hx, Ty) + D(hy, Tx)]\} < r + s$$

implies

$$H(Tx, Ty) < r. \qquad \text{.. (IV)}$$

Since $\{d(hx_n, hx_{n+1})\} \downarrow r$ there exists a positive integer N such that

$$r \leqslant d(hx_n, hx_{n+1}) < r + s \text{ for all } n > N. \qquad \text{...(V)}$$

Noting that

$$\text{Max}\{d(hx_n, hx_{n+1}), D(hx_n, Tx_n), D(hx_{n+1}, Tx_{n+1}),$$

$$\tfrac{1}{2}[D(hx_n, Tx_{n+1}) + D(hx_{n+1}, Tx_n)]\}$$

$$= d(hx_n, hx_{n+1}).$$

By (IV) and (V), $H(Tx_n, Tx_{n+1}) < r$.

By (III), $d(hx_{n+1}, hx_{n+2}) \leqslant H(Tx_n, Tx_{n+1}) < r$ which is a contradiction. Hence

$$r = 0 \text{ and } \{d(hx_n, hx_{n+1})\} \downarrow 0.$$

We now claim that $\{hx_n\}$ is a Cauchy sequence. Suppose not. Then there exists an $\epsilon > 0$ and a subsequence $\{hx_{n_i}\}$ of $\{hx_n\}$ such that $d(hx_{n_i}, hx_{n_{i+1}}) \geqslant 2\epsilon$.

Choose a number δ, $0 < \delta < \epsilon$ for which (I′) is satisfied. Since $\{d(hx_n, hx_{n+1})\} \downarrow 0$, there exists an Integer $N = N(\delta)$ such that

$$d(hx_i, hx_{i+1}) < \frac{\delta}{6} \text{ for all } i \geqslant N.$$

Let $n_i \geqslant N$. We shall show that there exists an integer $j \in (n_i, n_{i+1})$ such that

$$\epsilon + \frac{\delta}{3} \leqslant d(hx_{n_i}, hx_j) < \epsilon + \frac{2}{3}\delta. \qquad \ldots\text{(VI)}$$

We observe that

$$d(hx_{n_i}, hx_{n_{i+1}-1}) \geqslant \epsilon + \frac{\delta}{3}.$$

For, suppose that

$$d(hx_{n_i}, hx_{n_{i+1}} - 1) < \epsilon + \frac{\delta}{3}.$$

Then

$$d(hx_{n_i}, hx_{n_{i+1}}) \leqslant d(hx_{n_i}, hx_{n_{i+1}-1}) + d(hx_{n_{i+1}-1}, hx_{n_{i+1}}) < \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < 2\epsilon$$

a contradiction.

Pick j to be the smallest integer greater than n_i such that

$$d(hx_{n_i}, hx_j) \geqslant \epsilon + \frac{\delta}{3}.$$

Then

$$d(hx_{n_i}, hx_j) \leqslant d(hx_{n_i}, hx_{j-1}) + d(hx_{j-1}, hx_j) < \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \frac{2}{3}\delta.$$

Thus (VI) is established.

Noting that

$$\text{Max}\,\{d(hx_{n_i}, hx_j), D(hx_{n_i}, Tx_{n_i}), D(hx_j, Tx_j) + \frac{1}{2}[D(hx_{n_i}, Tx_j) + D(hx_j, Tx_{n_i})]\} < \epsilon + \delta$$

by (VI) and (I), we have

$$H(Tx_{n_i}, Tx_j) < \epsilon.$$

Now by (III),

$$d(hx_{n_i+1}, hx_{j+1}) \leqslant H(Tx_{n_i}, Tx_j) < \epsilon.$$

So,

$$d(hx_{n_i}, hx_j) \leqslant d(hx_{n_i}, hx_{n_i+1}) + d(hx_{n_i+1}, hx_{j+1}) + d(hx_{j+1}, hx_j)$$

$$< \frac{\delta}{6} + \epsilon + \frac{\delta}{6} = \epsilon + \frac{\delta}{3}, \text{ a contradiction to (VI)'}$$

Hence $\{hx_n\}$ is a Cauchy sequence. Since X is (T, h, x) orbitally complete, there exists a $z \in X$ such that $hx_n \to z$ as $n \to \infty$. Since $hx_{n+1} \in Tx_n$, $hhx_{n+1} \in hTx_n \subseteq Thx_n$, h is continuous on $\overline{h(X)}$ and T is continuous on $A = \overline{\bigcup_{a \in X} T(a)}$, we have

$$D(hz, Tz) \leqslant d(hz, hhx_{n+1}) + D(hhx_{n+1}, Tz)$$

$$\leqslant d(hz, hhx_{n+1}) + H(Thx_n, Tz).$$

Letting $n \to \infty$, $D(hz, Tz) \leqslant 0$ which implies that $hz \in Tz$. Thus the theorem is proved.

Theorem 2.2—Assume all the conditions in Theorem 2.1 and further assume that whenever $hx \in Ty$,

$$d(hx, hu) \leqslant H(Thy, Tu) \text{ for all } u \in X \qquad \text{...(A)}$$

then h and T have a unique common fixed point in X.

PROOF: From Theorem 2.1, there exists a $p \in X$ such that $hp \in Tp$.

Denote hp by z. By (A), $d(hp, hz) \leqslant H(Tz, Tz) = 0$.

Hence $hz = z$. $hz \subset hTp \subseteq Thp = Tz$.

Thus $z = hz \in Tz$.

Suppose $z' \neq z$ is such that $z' = hz' \in Tz'$

Since $z \in Tz$, $hz \in hTz \subseteq Thz$, by (A), Theorem 2.2, we have

$$d(hz, hz') \leqslant H(Thz, Tz').$$

That is,

$$d(z, z') = d(hz, hz') \leqslant H(Tz, Tz').$$

But by (I),

$$H(Tz, Tz') < \text{Max}\,\{d(hz, hz'), D(hz, Tz), D(hz', Tz'),$$
$$\tfrac{1}{2}[D(hz, Tz') + D(hz', Tz)]\}$$
$$\leqslant d(hz, hz') = d(x, z').$$

Thus $d(z\ \ z') < d(z, z\ )$ which is a contradiction.

Hence $z = z'$. Thus Theorem 2.2 is proved. Selecting T and h suitably we get generalizations of the previous results[1-5].

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