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A NOTE ON ABSOLUTE SUMMABILITY FACTORS OF INFINITE SERIES

G. D. DIKSHIT

Department of Mathematics and Statistics, University of Auckland
Private Bag, Auckland, New Zealand

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The main purpose of this note is to prove a theorem on the absolute summability factors for Cesàro summability of series, which improves upon a previous result of the author [Dikshit, G. D., *Proc. Natn. Inst. Sci. India* 25 A (1959) 191] and also rectifies some deficiencies in the proof of that result. Some new results on absolute Riesz summability factors of Fourier series are also given.

§1. Let Σu_n be a given series. The sequence $\{\sigma_n^\alpha\}$ of the Cesàro-means of order α, $\alpha > -1$, of Σu_n is defined by

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^\alpha u_\nu$$

where for any real ρ, $\{A_n^\rho\}$ is determined by the identity ;

$$\sum_{n=0}^{\infty} A_n^\rho x^n = (1-x)^{-\rho-1} \quad (|x| < 1).$$

We also define

$$A_{-n}^\rho = 0 \text{ for } n = 1, 2, \ldots .$$

For a sequence $\{\lambda_n\}$ we define

$$\Delta^\rho \lambda_n = \sum_{m=0}^{\infty} A_m^{-\rho-1} \lambda_{m+n}$$

whenever the series on the right is convergent.

When ρ is a non-negative integer, we get

$$\Delta^{\rho} \lambda_n = \sum_{m=0}^{\rho} A_m^{-\rho-1} \lambda_{m+n}.$$

Thus

$$\Delta^0 \lambda_n = \lambda_n \text{ and } \Delta^1 \lambda_n = \lambda_n - \lambda_{n+1}.$$

We write $\Delta\lambda_n$ for $\Delta^1 \lambda_n$. For positive integral values of h and k we obtain the formulae

$$\Delta^h (\Delta^k \lambda_n) = \Delta^{h+k} \lambda_n$$

and

$$\Delta^h (\lambda_n \mu_n) = \sum_{s=0}^{h} \binom{h}{s} \Delta^s \lambda_n \Delta^{h-s} \mu_{n+s}.$$

For a double sequence $\{\theta_{\nu,n}\}$, we write

$$\Delta_\nu (\theta_{\nu,n}) = \theta_{\nu,n} - \theta_{\nu+1,n}.$$

The series Σu_n is said to be absolutely summable (C, α) or summable $| C, \alpha |$, if the sequence $\{\sigma_n^{\alpha}\}$ is of bounded variation, that is to say

$$\sum | \sigma_n^{\alpha} - \sigma_{n-1}^{\alpha} |$$

is convergent.

Let $l = \{l_n\}$ be a positive, steadily increasing sequence tending to infinity with n. The series Σu_n is said to be summable by the absolute Riesz summability method $| R, l, \alpha |$ of 'type' l and 'order' α, $\alpha \geqslant 0$, if

$$\sum_{l(n) < \omega} \left\{1 - \frac{l(n)}{\omega}\right\}^{\alpha} u_n$$

is of bounded variation over (A, ∞), where A is a finite positive number[18].

In this paper we shall specifically refer to the Riesz method $| R, \exp(n^{\delta}), \alpha |$. By a result on 'second theorem of consistency' for absolute Riesz method[17] it is now known that for $\delta' > \delta > 0$, and $\alpha \geqslant 0$,

$$| R, \exp(n^{\delta'}), \alpha | \subset | R, \exp(n^{\delta}), \alpha | \subset | C, \alpha | .$$

It is also familiar (Dikshit[10]) that for $0 < \delta \leqslant 1$

$$| R, \exp(n^{\delta}), 1 | \subset | C, 1 - \delta |$$

and that for $\delta \geqslant 1$, and $\alpha \geqslant 0$, the method $| R, \exp(n^{\delta}), \alpha |$ is ineffective in the sense that only absolutely convergent series are summable by the method[9].

§2. We shall first prove the following

Theorem 1—Let $\alpha \geqslant 0$ and h be an integer such that $\alpha \leqslant h < (\alpha + 1)$. Let Σu_n be a given infinite series and $\{\sigma_n^{\alpha}\}$ be the sequence of its Cesàro-means of order α. Suppose $\{\mu_n\}$ is a positive and non-decreasing sequence and that

$$\sum_{\nu=1}^{n} | \sigma_{\nu}^{\alpha} - \sigma_{\nu-1}^{\alpha} | = O(\mu_n).$$

If $\{\lambda_n\}$ is such that $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy :

(i) $\mu_n \lambda_n = O(1)$

and

(ii) $\Sigma n^h \mu_n | \Delta^{h+1} \lambda_n | < \infty$

then $\Sigma u_n \lambda_n$ is summable $| C, \alpha |$.

Remarks : 1 This theorem with a further restriction on $\{\lambda_n\}$ in the case when α is non-integral, was given earlier by the author[8]. The proof of the theorem over there was so mewhat deficient as it depends on Lemma 3 of that paper, and the proof of Lemma 3 needed some more clarifications. The needful is done now as Lemma 4 below.

2. It may be noted that in the case $\mu_n = 1$, Theorem 1 furnishes sufficient conditions on $\{\lambda_n\}$ for $\Sigma u_n \lambda_n$ to be summable $| C, \alpha |$ whenever Σu_n is summable $| C, \alpha |$. Indeed, for this case much general and very sophisticated results with conditions on $\{\lambda_n\}$ which are both necessary and sufficient, are furnished in works of Chow[7], Peyerimhoff[10] and Bonanquet and Chow[4].

§3. We shall need the following lemmas for our proof.

Lemma 1 (Andersen[1])—If $\lambda_n = O(1)$ then for $\delta > k \geqslant -1$ and $> \delta 0$

$$\Delta^{\delta} \lambda_n = \sum_{m=n}^{\infty} A_{m-n}^{k-\delta} \Delta^{k+1} \lambda_m.$$

Lemma 2—For $\sigma > -1$ and $\sigma - \delta > 0$,

$$\sum_{\nu=n}^{\infty} A_{\nu-n}^{\delta} / (\nu A_{\nu}^{\sigma}) = 1/(n A_n^{\sigma-\delta-1}).$$

The result in this form is given by Chow[7]. I am thankful to the referee who has kindly pointed out that indeed it is a special case of a theorem of Gauss (see, for example, Bailey[22]).

Lemma 3 (Kuttner[16])—Let $\alpha > (s+1) > 0$ and s be non-integral. If $\{\lambda_n\}$ is such that $\{\Delta^s \lambda_n\}$ is well defined then $\Delta^s \lambda_n \to 0$ (C, α).

Lemma 4—Let p and q be real numbers and let $p > -1$. If $\{\mu_n\}$ is a positive non-decreasing sequence and $\{\lambda_n\}$ a bounded sequence such that

$$\Sigma\, n^p \mu_n \mid \Delta^{p+1} \lambda_n \mid < \infty \qquad \ldots(1)$$

then for $-1 < q \leqslant p$,

$$\Sigma\, n^q \mu_n \mid \Delta^{q+1} \mid < \infty \qquad \ldots(2)$$

and for $0 < q \leqslant p$,

$$\lim_{n \to \infty} n^q \mu_n \Delta^q \lambda_n = 0. \qquad \ldots(3)$$

PROOF OF LEMMA 4 : Note that for $\alpha > -1$,

$$A_n^{\alpha} \simeq \frac{n^{\alpha}}{\Gamma(\alpha+1)}.$$

Therefore, to prove (2) is the same as to prove that

$$\Sigma\, A_n^q \mu_n \mid \Delta^{q+1} \lambda_n \mid < \infty \qquad \ldots(2)'$$

when the condition (1) is replaced by

$$\Sigma\, A_n^p \mu_n \mid \Delta^{p+1} \lambda_n \mid < \infty. \qquad \ldots(1)'$$

First let us consider the case : $0 < p - q = \rho < 1$. We obtain that

$$\sum_{n=0}^{\infty} A_n^q \mu_n \mid \Delta^{q+1} \lambda_n \mid = \sum_{n=0}^{\infty} A_n^{p-\rho} \mu_n \mid \Delta^{p-\rho+1} \lambda_n \mid$$

$$\leqslant \sum_{n=0}^{\infty} A_n^{p-\rho} \mu_n \sum_{m=n}^{\infty} A_{m-n}^{\rho-1} \mid \Delta^{p+1} \lambda_m \mid$$

by Lemma 1,

$$= \sum_{m=0}^{\infty} \mid \Delta^{p+1} \lambda_m \mid \sum_{n=0}^{m} A_{m-n}^{\rho-1} A_n^{p-\rho} \mu_n$$

$$\leqslant \sum_{m=0}^{\infty} \mu_m \mid \Delta^{p+1} \lambda_m \mid \sum_{n=0}^{m} A_{m-n}^{\rho-1} A_n^{p-\rho}$$

$$= \sum_{0}^{\infty} A_m^p \mu_m \mid \Delta^{p+1} \lambda_m \mid .$$

Now (2)′ follows by successive iterations, as necessary.

To prove (3) we note that if

$$\lim_{n \to \infty} \Delta^q \lambda_n = 0,$$

then, as by Lemma 1,

$$\Delta^{q+1} \lambda_n = \Delta (\Delta^q \lambda_n)$$

$$= \Delta^q \lambda_n - \Delta^q \lambda_{n+1}$$

we obtain that

$$\Delta^q \lambda_n = \Delta^q \lambda_m + \sum_{k=n}^{m-1} \Delta^{q+1} \lambda_k$$

and, making $m \to \infty$, that $\Delta^q \lambda_n = \sum_{k=n}^{\infty} \Delta^{q+1} \lambda_k$.

Hence, in view of (2), for $p \geqslant q > 0$

$$n^q \mu_n | \Delta^q \lambda_n | = n^q \mu_n \sum_{k=n}^{\infty} \Delta^{q+1} \lambda_k$$

$$\leqslant \sum_{n}^{\infty} k^q \mu_k | \Delta^{q+1} \lambda_k$$

$$= o(1), \text{ as } n \to \infty.$$

Thus to complete the proof of the lemma we need to show that for

$$p \geqslant q > 0, \lim_{n\to\infty} \Delta^q \lambda_n = 0. \qquad \ldots(4)$$

In the case $p \geqslant q \geqslant 1$, by (2) we have

$$\Sigma n^{q-1} \mu_n | \Delta^q \lambda_n | < \infty$$

and thus

$$n^{q-1} \mu_n | \Delta^q \lambda_n | \to 0, \text{ as } n \to \infty.$$

As $\{n^{q-1} \mu_n\}$ is a non-decreasing positive sequence, we must have

$$\lim_{n\to\infty} \Delta^q \lambda_n = 0.$$

In the case $0 < q \leqslant p < 1$, as $\{n^q \mu_n\}$ is increasing

$$\Sigma | \Delta^{q+1} \lambda_n | \leqslant K \Sigma n^q \mu_n | \Delta^{q+1} \lambda_n | < \infty$$

that is,

$$\Sigma (\Delta^{q+1} \lambda_n) = \Sigma (\Delta^q \lambda_n - \Delta^q \lambda_{n+1})$$

is absolutely convergent and hence convergent. Therefore $\Delta^q \lambda_n$ tends to a limit as $n \to \infty$. However, by Lemma 3, we note that it cannot tend to a non-zero limit. Therefore, even in this case

$$\lim_{n\to\infty} \Delta^q \lambda_n = 0,$$

and thus (4) is completely proved. This completes the proof of the lemma.

§4. *Proof of Theorem* 1—Let

$$t_n^\alpha = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\, \nu u_\nu \quad \text{and} \quad \tau_n^\alpha = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha}\, u_\nu\, \lambda_\nu.$$

Then

$$\sigma_n^\alpha - \sigma_{n-1}^\alpha = \frac{1}{n\, A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\, \nu u_\nu = \frac{t_n^\alpha}{n\, A_n^\alpha}.$$

Now

$$\tau_n^\alpha = \frac{\lambda_n}{A_n^\alpha} \sum_{0}^{n} A_{n-\nu}^{\alpha} u_\nu + \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha} (\lambda_\nu - \lambda_n)\, u_\nu$$

$$= \xi_n^\alpha + \zeta_n^\alpha, \text{ say.}$$

Thus, to prove the theorem, it is sufficient to show that

$$\Sigma\, |\, \Delta \xi_n^\alpha\, | < \infty \qquad \ldots(5)$$

and that

$$\Sigma\, |\, \Delta \zeta_n^\alpha\, | < \infty. \qquad \ldots(6)$$

PROOF OF (15): Note that

$$\Sigma\, |\, \Delta \xi_n^\alpha\, | = \Sigma\, |\, \Delta (\lambda_n\, \sigma_n^\alpha)\, |$$

$$\leqslant \Sigma\, |\, \sigma_n^\alpha\, |\, |\, \Delta \lambda_n\, | + \Sigma\, |\, \lambda_{n+1}\, |\, |\, \Delta \sigma_n^\alpha\, |.$$

Since

$$|\, \sigma_n^\alpha - \sigma_0^\alpha\, | = \left| \sum_{\nu=1}^{n} (\sigma_\nu^\alpha - \sigma_{\nu-1}^\alpha) \right| \leqslant \sum_{\nu=1}^{n} \left| \sigma_\nu^\alpha - \sigma_{\nu-1}^\alpha \right|$$

we obtain

$$\sigma_n^{\alpha} = O(\mu_n).$$

We also note that the hypothesis (i) gives that $\{\lambda_n\}$ is a bounded sequence. Therefore by Lemma 4,

$$\sum \left|\sigma_n^{\alpha}\right| \, |\Delta\lambda_n| \leqslant K \Sigma \mu_n |\Delta\lambda_n| < \infty. \qquad \ldots(7)$$

Also

$$\sum_{\nu=1}^{n} \left|\lambda_{\nu+1}\right| |\Delta\sigma_{\nu}^{\alpha}| \leqslant \sum_{\nu=1}^{n-1} |\Delta\lambda_{\nu+1}| \left(\sum_{1}^{\nu} \left|\Delta\sigma_m^{\alpha}\right|\right) + |\lambda_{n+1}| \sum_{1}^{n} |\Delta\sigma_{\nu}^{\alpha}|$$

$$\leqslant K\left(\sum_{1}^{n} \mu_{\nu+1} |\Delta\lambda_{\nu+1}| + \mu_{n+1} |\lambda_{n+1}|\right)$$

and therefore again by Lemma 4,

$$\sum |\lambda_{n+1}| \, |\Delta\sigma_n^{\alpha}| < \infty$$

and this proves (5).

PROOF OF (6): For $n \geqslant 1$

$$\zeta_n^{\alpha} - \zeta_{n-1}^{\alpha} = \frac{1}{A_n^{\alpha}} \sum_{0}^{n} A_{n-\nu}^{\alpha} (\lambda_\nu - \lambda_n) u_\nu - \frac{1}{A_{n-1}^{\alpha}} \sum_{0}^{n-1} A_{n-1-\nu}^{\alpha} (\lambda_\nu - \lambda_{n-1}) u_\nu$$

$$= \sum_{0}^{n-1} \left(\frac{A_{n-\nu}^{\alpha}}{A_n^{\alpha}} - \frac{A_{n-1-\nu}^{\alpha}}{A_{n-1}^{\alpha}}\right) (\lambda_\nu - \lambda_n) u_\nu + (\lambda_{n-1} - \lambda_n) \frac{1}{A_{n-1}^{\alpha}} \times \sum_{0}^{n-1} A_{n-1-\nu}^{\alpha} u_\nu$$

$$= \frac{1}{n A_n^{\alpha}} \sum_{1}^{n-1} \theta_{\nu,n} \nu u_\nu + \sigma_{n-1}^{\alpha} \Delta\lambda_{n-1}, \text{ say,} \qquad \ldots(8)$$

where

$$\theta_{\nu,n} = A_{n-\nu}^{\alpha-1}(\lambda_\nu - \lambda_n).$$

We have already shown in (7) that $\Sigma_1 \sigma_n^\alpha \Delta\lambda_n| < \infty$. Applying Abel's transformation h times we obtain thar for $n \geqslant 2$

$$\sum_{\nu=1}^{n-1} \nu u_\nu \theta_{\nu,n} = \sum_{\nu=1}^{n-1} t_\nu^h \Delta_\nu^h \theta_{\nu,n} \qquad \ldots(9)$$

where

$$\Delta_\nu^h \theta_{\nu,n} = \sum_{s=1}^{h} \binom{h}{s} \Delta_\nu^{h-s} A_{n-\nu-s}^{\alpha-1} \Delta^s \lambda_\nu + (\lambda_\nu - \lambda_n) \Delta_\nu^h A_{n-\nu}^{\alpha-1}. \qquad \ldots(10)$$

Therefore, after (8), (7), (9) and (10), to prove (6) it is enough to prove that for each s, $1 \leqslant s \leqslant h$,

$$S_1 = \sum_{n=2}^{\infty} \frac{1}{nA_n^\alpha} \left| \sum_{\nu=1}^{n-1} t_\nu^h \Delta^s \lambda_\nu \Delta_\nu^{h-s} A_{n-\nu-s}^{\alpha-1} \right| < \infty \qquad \ldots(11)$$

and that

$$S_2 = \sum_{n=2}^{\infty} \frac{1}{nA_n^\alpha} \left| \sum_{\nu=1}^{n-1} t_\nu^h (\lambda_\nu - \lambda_n) \Delta_\nu^h A_{n-\nu}^{\alpha-1} \right| < \infty. \qquad \ldots(12)$$

PROOF OF (11): First note that

$$t_\nu^h = \sum_{k=1}^{\nu} A_{\nu-k}^{h-\alpha-1} t_k^\alpha$$

and therefore

$$\sum_1^n | \sigma_\nu^h - \sigma_{\nu-1}^h | = \sum_1^n \frac{| t_\nu^h |}{\nu A_\nu^h}$$

(equation continued on p. 1335)

$$= \sum_{1}^{n} \frac{1}{\nu A_\nu^h} \left| \sum_{k=1}^{\nu} A_{\nu-k}^{h-\alpha-1} t_k^\alpha \right|$$

$$\leqslant \sum_{1}^{n} | t_k^\alpha | \sum_{\nu=k}^{n} \frac{A_{\nu-k}^{h-\alpha-1}}{\nu A_\nu^h}$$

$$\leqslant \sum_{1}^{n} \frac{| t_k^\alpha |}{k A_k^\alpha}, \text{ by Lemma 2,}$$

$$= \sum_{1}^{n} | \sigma_k^\alpha - \sigma_{k-1}^\alpha | . \qquad \ldots(13)$$

Now, since

$$\Delta_\nu^{h-s} A_{n-\nu-s}^{\alpha-1} = A_{n-\nu-s}^{\alpha-1-h+s} \text{ and } A_{-m}^{\alpha-1-h+s} = 0, \text{ for } m = 1, 2, \ldots,$$

we obtain that for each s, $1 \leqslant s \leqslant h$,

$$S_1 = \sum_{n=1}^{\infty} \frac{1}{n A_n^\alpha} \left| \sum_{\nu=1}^{n-s} t_\nu^h A_{n-\nu-s}^{\alpha-1-h+s} \Delta^s \lambda_\nu \right|$$

$$\leqslant \sum_{\nu=1}^{\infty} | t_\nu^h | | \Delta^s \lambda_\nu | \sum_{n=\nu+s}^{\infty} \frac{A_{n-\nu-s}^{\alpha-1-h+s}}{n A_n^\alpha}$$

$$= \sum_{\nu=1}^{\infty} | t_\nu^h | | \Delta^s \lambda_\nu | \frac{1}{(\nu+s) A_{\nu+s}^{h-s}}, \text{ by Lemma 2,}$$

$$= \sum_{\nu=1}^{\infty} \nu A_\nu^h | \sigma_\nu^h - \sigma_{\nu-1}^h | | \Delta^s \lambda_\nu | \frac{1}{(\nu+s) A_{\nu+s}^{h-s}}$$

$$\leqslant K \sum_{\nu=1}^{\infty} \nu^s | \Delta^s \lambda_\nu | | \sigma_\nu^h - \sigma_{\nu-1}^h |$$

$$\leqslant K \sum_{\nu=1}^{\infty} \{\nu^{s-1} \mid \Delta^s \lambda_\nu \mid + \nu^s \mid \Delta^{s+1} \lambda_\nu \mid \} \mu_\nu + \lim_{N \to \infty} N^s \mu_N \mid \Delta^s \lambda_N \mid ,$$

by (13),

$< \infty$, by Lemma 4.

PROOF OF (12) : In the case when α is an integer, that is when $\alpha = h$, we note that

$$\Delta_\nu^h \; A_{n-\nu}^{\alpha-1} = A_{n-\nu}^{-1} = \begin{cases} 1, & \text{for } n = \nu \\ 0, & n \neq \nu \end{cases}$$

and therefore

$$(\lambda_\nu - \lambda_n) \; \Delta_\nu^h \; A_{n-\nu}^{\alpha-1} = 0$$

for every $\nu \leqslant n$. Thus

$$S_2 = 0.$$

In the case when α is non-integral : $\alpha < h < (\alpha + 1)$, we write

$$\lambda_\nu - \lambda_n = \sum_{\rho=\nu}^{n-1} \Delta\lambda_\rho$$

and also note that

$$A_{n-\nu}^{\alpha-h-1} \; A_{\nu-k}^{h-\alpha-1} = \left(\frac{h-\alpha}{n-k}\right) \Delta_\nu \left(A_{n-\nu}^{\alpha-h} \; A_{\nu-k-1}^{h-\alpha} \right).$$

Thus

$$\sum_{\nu=1}^{n-1} t_\nu^h \, (\lambda_\nu - \lambda_n) \; (\Delta_\nu^h \; A_{n-\nu}^{\alpha-1}) = \sum_{\nu=1}^{n-1} A_{n-\nu}^{\alpha-h-1} \left(\sum_{k=1}^{\nu} A_{\nu-k}^{h-\alpha-1} \, t_k^\alpha \right) \left(\sum_{\rho=\nu}^{n-1} \Delta\lambda_\rho \right)$$

$$= \sum_{k=1}^{n-1} t_k^\alpha \sum_{\rho=k}^{n-1} (\Delta\lambda_\rho) \sum_{\nu=k}^{\rho} A_{n-\nu}^{\alpha-h-1} \; A_{\nu-k}^{h-\alpha-1}$$

$$= \sum_{k=1}^{n-1} t_k^\alpha \sum_{\rho=k}^{n-1} (\Delta\lambda_\rho) \sum_{\nu=k}^{\rho} \left(\frac{h-\alpha}{n-k}\right) \Delta_\nu \left(A_{n-\nu}^{\alpha-h} A_{\nu-k-1}^{h-\alpha} \right)$$

(equation continued on p. 1337)

$$= -(h-\alpha)\sum_{k=1}^{n-1} t_k^{\alpha}\frac{1}{(n-k)}\sum_{\rho=k}^{n-1}(\Delta\lambda_\rho)$$

$$\times\left(A_{n-\rho-1}^{\alpha-h}\ A_{\rho-k}^{h-\alpha}\right).$$

Hence

$$S_2 \leqslant (h-\alpha)\sum_{n=2}^{\infty}\frac{1}{nA_n^{\alpha}}\sum_{k=1}^{n-1}\frac{|\,t_k^{\alpha}\,|}{(n-k)}\sum_{\rho=k}^{n-1}A_{n-1-\rho}^{\alpha-h}\ A_{\rho-0}^{h-\alpha}\,|\,\Delta\lambda_\rho\,|$$

$$= (h-\alpha)\sum_{\rho=1}^{\infty}|\,\Delta\lambda_\rho\,|\sum_{k=1}^{\rho}A_{\rho-k}^{h-\alpha}\left|t_k^{\alpha}\right|\sum_{n=\rho+1}^{\infty}\frac{A_{n-1-\rho}^{\alpha-h}}{n\,(n-k)\,A_n^{\alpha}}. \qquad \ldots(14)$$

However, note that for $\rho > k$,

$$A_{\rho-k}^{h-\alpha}\sum_{n=\rho+1}^{\infty}\frac{A_{n-1-\rho}^{\alpha-h}}{n\,(n-k)\,A_n^{\alpha}} = A_{\rho-k}^{h-\alpha}\left(\sum_{n=\rho+1}^{2\rho+1-k}+\sum_{2\rho+2\ k}^{\infty}\right)\frac{A_{n-1-\rho}^{\alpha-h}}{n\,(n-k)\,A_n^{\alpha}}$$

$$\leqslant \frac{A_{\rho-k}^{h-\alpha}}{\rho\,(\rho+1-k)\,A_{\rho}^{\alpha}}\sum_{n=\rho+1}^{2\rho+1-k}A_{n-1-\rho}^{\alpha-h}$$

$$+\frac{A_{\rho-k}^{h-\alpha}}{\rho A_{\rho}^{\alpha}}\sum_{n=2\rho+2-k}^{\infty}\frac{A_{n-1-\rho}^{\alpha-h}}{(n-\rho)}$$

$$\leqslant O\left(\frac{1}{\rho A_{\rho}^{\alpha}}\right)$$

$$= O\left(\frac{1}{kA_k^{\alpha}}\right). \qquad \ldots(15)$$

Therefore from (14) and (15),

$$S_2 \leqslant K\sum_{\rho=1}^{\infty}|\,\Delta\lambda_\rho\,|\sum_{k=1}^{\rho}\frac{|\,t_k^{\alpha}\,|}{kA_k^{\alpha}}$$

(*equation continued on p.* 1338)

$$\leqslant K \sum_{p=1}^{\infty} \mu_p \quad \Delta\lambda_p \mid < \infty.$$

This completes the proof of the theorem.

§5. In this section we obtain some results on absolute summability of a Fourier series as an application of Theorem 1.

5.1. Let $f \in L(-\pi, \pi)$ and be a periodic function with period 2π and let

$$f(t) \sim \frac{1}{2} a_0 + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \Sigma A_n(t).$$

Let x be fixed and let us write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

$$\phi_\alpha(t) = \alpha t^{-\alpha} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \quad \alpha > 0$$

and

$$\phi_0(t) = \phi(t).$$

5.2. We prove the following theorems.

Theorem 2—Let $\alpha \geqslant 0$ and h be an integer such that $\alpha \leqslant h < (\alpha + 1)$. Let $\{\epsilon_n\}$ be such that

(i) $\epsilon_n \log n = O(1)$ and (ii) $\Sigma n^h (\log n) \mid \Delta^{h+1} \epsilon_n \mid < \infty$.

If $\phi_\alpha(t) \in BV(0, \pi)$ then $\Sigma A_n(x) \epsilon_n$ is summable $\mid C, \alpha \mid$.

Theorem 3—Let $0 < \delta \leqslant 1$ and numbers α and h and the sequence $\{\epsilon_n\}$ be as in Theorem 2. If $\phi_\alpha(t) \in BV(0, \pi)$ then $\Sigma A_n(x) \epsilon_n n^{-\alpha\delta}$ is summable $\mid R, \exp(n^\delta), \alpha \mid$.

5.3. *Remarks*: (1) Theorem 2 is an improvement on the corresponding result given earlier (see Dikshit[5]). With $\epsilon_n = (\log n)^{-1-\eta}$, $\eta > 0$, Theorem 2 has initially been given by Cheng[6] for $0 \leqslant \alpha \leqslant 1$. Sunouchi[21] extended Cheng's result to all values of α, $\alpha \geqslant 0$.

(2) Quite recently the summability $\mid R, \exp(n^\delta), \alpha \mid$, $0 < \delta < 1$, $\alpha \geqslant 0$, of a Fourier series has been studied rather extensively in another paper[13]. As a special case of the main theorem therein[13] one obtains that for $\alpha \geqslant 0$ and $c > \alpha\delta$, $\phi_\alpha(t) \in BV(0, \pi)$ implies $\Sigma A_n(x) n^{-c}$ is summable $\mid R, \exp(n^\delta), \alpha \mid$ (see Dikshit[13], Corollary 2). Thus

Theorem 3 improves upon this result substantially. Results in Theorem 3 are comparable with existing ones. For example, consider the case $\alpha = 1$. By Theorem 2 and the well-known Kogbetliantz result on $| C |$-summability factors, we get that under the hypothesis of Theorem 3 on f, $\Sigma A_n(x)\,\epsilon_n\, n^{-\delta}$ is summable $| C, 1-\delta |$, whereas Theorem 3 gives us that $\Sigma A_n(x)\,\epsilon_n\, n^{-\delta}$ is summable $| R, \exp(n^{\delta}), 1 |$, which, in view of the inclusion relations between $| C |$ and $| R, \exp(n^{\delta}), 1 |$ is a better result. However, it appears that in some instances, the results obtain by the present techniques may not be as refined as the results one may obtained by a more direct approach. For example, a classical result due to Bosanquet[3,4] is that for $\gamma > \alpha \geqslant 0$, $\phi_{\alpha}(t) \in BV(0, \pi)$ implies that $\Sigma A_n(x)$ is summable $| C, \gamma |$, and hence by Lemma 6 (given below) we get that for $\gamma > \alpha \geqslant 0$, $\beta \geqslant 0$, $\phi_{\alpha}(t) \in BV(0, \pi)$ implies that (i) $\sum_{2}^{\infty} A_n(x)\,(\log n)^{-\gamma\beta}$ is summable $| R, \exp((\log n)^{\beta+1}), \gamma |$, and (ii) $\Sigma A_n(x) n^{-\gamma\delta}$ is summable $| R, \exp(n^{\delta}), \gamma |$. However working directly we have the results that for $\gamma > \alpha > 0$ and $\beta \geqslant 0$, $\phi_{\alpha}(t) \in BV(0, \pi)$ implies that (i) $\sum_{2}^{\infty} A_n(x)\,(\log n)^{\alpha-\beta}$ is summable $| R, \exp((\log n)^{\beta}), \gamma |$, and (ii) $\Sigma A_n(x)\, n^{-\alpha\delta}$ is summable $| R, \exp(n^{\delta}), \gamma |$. (For, $0 < \alpha < 1$, the result (i) is given in Dikshit[11]. Recently[15] the result has been extended to hold for all values of $\alpha : \gamma > \alpha > 0$, and a version for $\alpha = 0$ also is given in Dikshit and Rees[14]. The result for the summability $| R, \exp(n^{\delta}), \gamma |$ is given in Dikshit[13]).

(3) Results similar to Theorem 3 but involving Riesz summability of other 'types' e.g., those determined by (i) $\exp((\log n)^{\beta})$, $\beta > 1$, (ii) $\exp(n^{\delta}/(\log n)^{\beta})$, $0 < \delta \leqslant 1$, $\beta > 0$, etc., may also be obtained.

5.4. *Lemmas*—We shall further need the following results to prove Theorems 2 and 3.

Lemma 5—Let $\alpha \geqslant 0$ and $\sigma_n^{\alpha}(t)$ be the n-th Cesàro mean of order α of the Fourier series $\Sigma A_n(t)$. If $\phi_{\alpha}(t) \in BV(0, \pi)$ then

$$\sigma_n^{\alpha}(x) - \sigma_{n-1}^{\alpha}(x) = O(1/n), \quad \text{as } n \to \infty.$$

This result is given independently by Prasad and Bhatt[20] and Chen[5] (*Cf.* also Sunouchi[21]).

Lemma 6—Let $0 < \delta \leqslant 1$, $\beta \geqslant 0$ and $\alpha \geqslant 0$. If Σu_n is summable $| C, \alpha |$ then

(i) $\Sigma u_n n^{-\alpha\delta}$ is summable $| R, \exp(n^{\delta}), \alpha |$,

and

(ii) $\Sigma u_n (\log n)^{-\alpha\beta}$ is summable $| R, \exp((\log n)^{\beta+1}), \alpha |$.

This is a special case of a general result proved in Dikshit[12].

5.5. PROOF OF THEOREMS 2 AND 3 :

Theorem 2 follows from Lemma 5 and Theorem 1. Similarly Theorem 3 follows from Theorem 2 and Lemma 6.

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REFERENCES

1. A. F. Andersen, *Proc. Lond. Math. Soc.* (2) **27** (1928) 39-71.
2. L. S. Bosanquet, *J. Lond. Math. Soc.* **11** (1936) 11-15.
3. L. S. Bosanquet, *Proc. Lond. Math. Soc.* (2) **41** (1936) 517-28.
4. L. S. Bosanquet, and H. C. Chow, *J. Lond. Math. Soc.* **32** (1957) 73-82.
5. K. K. Chen, *Summation of the Fourier series of Orthogonal Fnnctions*, Chapter IV Science Press, Peking (1957).
6. M. T. Cheng, *Duke Math. J.* **15** (1948) 29-36.
7. H. C. Chow, *J. Lond. Math. Soc.* **29** (1954) 459-76.
8. G. D. Dikshit, *Proc. Natn. Inst Sci. India* **25A** (1959) 191-200.
9. G. D. Dikshit, *Proc. Natn. Inst. Sci. India* **26A** (1960) 541-44.
10. G. D. Dikshit, *Indian J. Math.* **7** (1965) 73-78.
11. G. D. Dikshit, *Kyungpook Math. J.* **17** (1977) 195-201.
12. G. D. Dikshit, *Indian J. Math.* **25** (1983) 245-56.
13. G. D. Dikshit, *Analysis* **5** (1985) 1-13.
14. G. D. Dikshit, and C. S. Rees, *Proc. Am. Math. Soc.* **82** (1981) 231-38.
15. G. D. Dikshit, and C. S. Rees, *J. Math. Analysis and Applications* **102** (1984) 549-65.
16. B. Kuttner, *J. Lond. Math. Soc.* **43** (1968) 758-62.
17. B. Kuttner, *Proc. Lond. Math. Soc.* (3) **29** (1974) 17-32.
18. N. Obreschkoff, *C. R. Acad. Sci. Paris* **186** (1928) 215-17.
19. A. Peyerimhoff, *Math. Z.* **59** (1954) 417-24.
20. B. N. Prasad, and S. N. Bhatt, *Duke Math. J.* **24** (1957) 103-17.
21. G. Sunouchi, *Kodai Math. Sem. Rep.* **1** (1954) 59-62.
22. W. N. Bailey, Generalised Hypergeometric Series, *Cambridge Tract in Math.* No. **32** (1935) pp. 2-3.