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ON FIXED POINTS IN HAUSDORFF SPACES

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Some theorems are presented concerning the existence of fixed point of a contractive type mapping on a Hausdorff space.

Let (X, d) be a metric space. A mapping T of X into itself is said to be contractive if

$$d(Tx, Ty) < d(x, y), \forall \; x \neq y \in X.$$

Edelstein[3] proved the following theorem : Let (X, d) be a metric space, T a contractive self mapping of X. If for some $x \in X$ the sequence of iterates $(T^n x)$ has a convergent subsequence (Tx^{nk}) converging to a point $u \in X$, then $u = \lim (Tx^{nk})$ is a unique fixed point.

The purpose of this paper is to extend this result for mappings on Hausdorff spaces. See also Chatterjee[1], Popa[4].

Theorem 1—Let X be a Hausdorff space, $T : X \to X$ continuous function, and $F : X \times X \to R^+$ be a continuous function such that

(a) $F(x, y) \neq 0, \; \forall \; x \neq y$

(b) $F(Tx, Ty) \leqslant F(x, y) - \omega(F(x, y)), \forall \; x, y \in X$

where $\omega : R^+ \to R^+$ is continuous function, with $Q < \omega(r) < r$ for all $r \in R_+ - \{0\}$.

If for some $x_0 \in X$ the sequence $x_n = (T^n x_0)$ has a convergent subsequence, then T has a unique fixed point.

PROOF : By (b) we obtain

$$\sum_{k=0}^{n} \omega (F (Tx^k, Tx^{k+1})) \leqslant F (x, T_x), \forall\ n > 1, x \in X.$$

Thus the series

$$\sum_{k=0}^{\infty} \omega (F (Tx^k , Tx^{k+1}))$$

of non-negative terms is convergent.

From this result follows

$$\lim_{k \to \infty} \omega (F (Tx^k, \ Tx^{k+1})) = 0.$$

Since ω (0) $=$ 0, from the continuity of ω we have

$$\lim_{k \to \infty} \omega (F (Tx^k, Tx^{k+1})) = \omega (\lim_{k \to \infty} F (Tx^k , Tx^{k+1})) = 0$$

which implies $\lim\limits_{k \to \infty} F (Tx^k, \ Tx^{k+1})) = 0.$

For the point $x_0 \in X$ we have $\lim\limits_{k \to \infty} T^k x_0 = u$, where $(T^k \ x_0)$ is a convergent subsequence of (x_n), so we arrive at the relation

$$F (u, Tu) = F (\lim_{k \to \infty} T^k x_0, \lim_{k \to \infty} T^{k+1} x_0) = \lim_{k \to \infty} F (Tx^k_0 , Tx^{k+1}_0) = 0.$$

From (a) implies $u = Tu$.

If $v \neq u$ be another fixed point, then

$$F(u, v) = F(Tu, Tv) \leqq F(u, v) - \omega (F(u, v))$$

which is a contradiction. Hence T has a unique fixed point

Corollary 1—Let (X, d) be complete metric space. If $T : X \to X$ is a function such that

$$d (Tx, Ty) \leqslant d (x, y) - \omega (d (x, y)), \forall\ x, y \in X$$

where $\omega : R^+ \to R^+$ is continuous function, with $0 < \omega(r) < r$ for all $r \in R^+ - \{0\}$, then T has a unique fixed point.

PROOF : From the relation (Theorem 1)

$$\lim_{k \to \infty} d(Tx^k, Tx^{k+1}) = 0,\ x \in X$$

follows that the sequence (Tx^k) is Cauchy in X and therefore is convergent. We complete the proof as in Theorem 1.

Remark 1 : If X is compact space then the condition (b) in Theorem 1 can be weaked as follows[5].

(b′) for all $x \in X,\ x \neq Tx$ holds $F(Tx, T^2x) < F(x, Tx)$

where $F : X \times X \to R^+$ is a lower semi-continuous function.

Remark 2 : If X is metric space then in the Theorem 1 we can consider as F the metric d. But then the best result is the theorem of M. (Edelstein[3]) A result similar to Corollary 1 is given in Dugundji and Cranas[2].

Theorem 2—Let T_1 and T_2 be a continuous functions of a Hausdorff space X into itself and let $F : X \times X \to R^+$ be a continuous function such that

a) $F(x, y) \neq 0,\ \forall\ x \neq y$

b) $F(T_1x,\ T_2y) \leqslant F(x, y) - \omega(F(x, y)),\ \forall\ x \neq y \in X$

where $\omega : R^+ \to R^+$ is continuous function, with $0 < \omega(r) < r$ for all $r \in R^+ - \{0\}$.

If for some $x_0 \in X$ the sequence (x_n), where $T_1x_{2n} = x_{2n+1}$ and $T_2\, x_{2n+1} = x_{2n+2}$, $n = 0, 1, 2, \ldots$ has a convergent subsequence of the type $(x_{(2m+1)n})$, $m \in N$ is fixed, the T_1 and T_2 have a unique fixed point.

PROOF : For simplicity we suppose that (x_{3n}) is convergent to u. We consider the subsequence $(x_{3(2k)})$ which converge also to u.

From the continuity of T_1 and T_2 we have

$$T_1u = T_1 \lim x_{3(2k)} = \lim T_1\, x_{3(2k)} = \lim x_{3(2k+1)}$$

$$T_2\, T_1\, u = T_2 \lim x_{3(2k+1)} = \lim T_2\, x_{3(2k+1)} = \lim x_{3(2k+2)}.$$

Thus we have, as $k \to \infty$,

$$F(u, T_1 u) F = (\lim x_{3(2k)}, \lim x_{3(2k+1)})$$

$$= \lim F(x_{3(2k)}, x_{3(2k+1)}) = Z$$

$$= \lim F(x_{3(2k+1)}, x_{3(2k+2)}) = F(T_1 u, T_2 T_1 u)$$

where

$$z = \lim_{n \to \infty} F(x_n, x_{n+1}).$$

As in Theorem 1 we can prove that $z = 0$.

From (a) implies $u = T_1 u$ and $T_2 u = u$.

If $v \neq u$ be another fixed point of T_1 and T_2, then

$$F(u, v) = F(T_1 v, T_2 u) \leqslant F(u, v) - \omega(F(u, v))$$

which is a contradiction.

Hence T_1 and T_2 have a unique fixed point.

Corollary 2—Let (X, d) be metric space and $T_1 : X \to X$, $T_2 : X \to X$ be continuous functions such that

$$d(T_1 x, T_2 y) \leqslant d(x, y) - \omega(d(x, y)), \forall\, x \neq y \in X$$

where $\omega : R^+ \to R^+$ is continuous function, with $0 < \omega(r) < r$ for all $r \in R^+ - \{0\}$. If for some $x_0 \in X$ the sequence (x_n), where $T_1 x_{2n} = x_{2n+1}$ and $T_2 x_{2n+1} = x_{2n+2}$, $n = 0, 1, 2, \ldots$, has a convergent subsequence of the type (x_{3n}), then T_1 and T_2 have a unique fixed point

Remark 3 : If in Corollary 2 (X, d) is complete metric space then the sequence (x_n) is convergent for all $x_0 \in X$, because (Theorem 1)

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

Theorem 3— Let $T_1, T_2, \ldots, T_k$ be continuous functions of a Hausdorff space X into itself and let $F : X \times X \to R^+$ be a continuous function such that

a) $F(x, y) \neq 0, \forall\, x \neq y$

b) $F(T_i x, T_{i+1} y) \leqslant F(x, y) - \omega(F(x, y)), \forall\, x \neq y \in X$

where $\omega : R^+ \to R^+$ is continuous function, with $0 < \omega(r) < r$

for all $r \in R^+ - \{0\}$.

If for some $x_0 \in X$ the sequence (x_n), where

$$x_1 = T_1 x_0, x_2 = T_2 x_1, \ldots, x_k = T_k x_{k-1}$$

$$x_{k+1} = T_1 x_k, \; x_{k+2} = T_2 x_{k+1}, \ldots, x_{2k} = T_k x_{2k-1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{nk+1} = T_1 x_{nk}, \; x_{nk+2} = T_2 x_{nk+1}, \ldots, x_{(n+1)k} = T_k x_{(n+1)k-1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

for $n = 0,1,2,\ldots$ has a covergent subsequence of the type $(x_{(mk+1)n})$, where $m \in N$, then $T_1, T_2, \ldots, T_k$ have a unique fixed point.

PROOF : It is enough to observe that $F(x_n, x_{n+1}) < F(x_{n-1}, x_n)$, $n = 0,1,2,\ldots$ and $\lim_{n\to\infty} F(x_n, x_{n+1}) = 0$, and the proof follows as in Theorem 2.

Remark 4 : A corollary of the Theorem 3 similar to Corollary 2 can be formulated.

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