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ON THE MINIMUM OF A CLASS OF NONDIFFERENTIABLE FUNCTIONS

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A class of nondifferentiable functions that is relevant to nondifferentiable mathematical programming is introduced. A sufficient condition is given under which each function in the class attains its infimum on a polyhedral convex set if it is bounded below there. Quadratic functions, as a special case, are shown to satisfy this sufficient condition.

1. INTRODUCTION

It is well known[1] that a quadratic (not necessarily convex) function Q bounded below on a polyhedral convex set $X \subset R^n$ attains its infimum there. Perold[5] defines a class of continuous functions G which, in particular, contains all quadratic functions, and shows that each function in G attains its infimum on X if and only if it is bounded below on X.

In this paper we consider the class of functions $F : R^n \to R$ given by

$$F(x) = \frac{1}{2} x^T Dx + h(x)$$

where D is an $n \times n$ symmetric matrix and h is a lower semicontinuous sublinear function. Note that $h : X \to R$ is sublinear if and only if it satisfies

Property I : $h(x + y) \leqslant h(x) + h(y)$ for all $x, y \in X$.

Property II : $h(\lambda x) = \lambda h(x)$ for all $x \in X$ and all $\lambda \geqslant 0$.

Clearly, quadratic functions are contained in the above class. We give sufficient conditions under which the function F attains its infimum on X. An existence theorem to a generalized complementarity problem is obtained by using this result. The theorems given here are particularly relevant to the nondifferentiable optimization problems, such as those in Mond[2]. Mond and Schechter[3] and Schechter[8], where the objective function is the sum of a differentiable function and a lower semicontinuous positively homogeneous convex function. The latter function generally appears in any of the following forms:

(a) $(x^T Bx)^{1/2}$ with B as an $n \times n$ symmetric positive semi-definite matrix.

(b) $\|Sx\|_p$ $(p > 1)$ where S is a $k \times n$ matrix and the p norm is given by

$$\|y\|_p = \left(\sum_{i=1}^{k} | y_i |^p \right)^{1/p}.$$

(c) $\max (x^T v \mid v \in C)$, where C is any compact convex subset of R^n.

It can be easily checked that the functions (a), (b) and (c) are sublinear.

2. Main Results

Let X be a nonempty polyhedral convex set of the form

$$X = \{x \in R^n : Ax \leqslant b\}$$

where $A \in R^{m \times n}$. For real $\alpha > 0$, consider the set

$$X(\alpha) = \{x : x \in X, \|x\| \leqslant \alpha\}$$

where $\|x\|$ denotes Euclidean norm of x. Since X is nonempty, there exists α_0 such that the sets $X(\alpha)$, for $\alpha_0 \leqslant \alpha < \infty$, are nonempty and compact. The lower semicontinuous function F attains its infimum on each such compact set. Let y be a minimal point of F on $X(\alpha)$, and let

$$f(\alpha) = \inf_{x \in X(\alpha)} F(x). \qquad \ldots(1)$$

Then

$$F(y) = f(\alpha). \qquad \ldots(2)$$

Because $\alpha < \alpha'$ implies $X(\alpha) \subset X(\alpha')$, the function $f(\alpha)$ is monotone nonincreasing. Thus $f(\alpha)$ is bounded above for $\alpha_0 \leqslant \alpha < \infty$.

Lemma 1—Let F be bounded below on X. Let $\{\alpha_k\}$ be a sequence of positive reals with $\alpha_k \to \infty$. If, for every $\alpha \in \{\alpha_k\}$, there is a minimal point x^α of F on $X(\alpha)$ such that $\|x^\alpha\| = \alpha$, then there exists a vector t $(t \neq 0)$ satisfying

$$At \leqslant 0,\ t^T Dt = 0,\ h(t) \leqslant 0. \qquad \ldots(3)$$

Moreover, there is a fixed k and a scalar $\delta > 0$ such that

$$x^k - \lambda t \in X,\ \|x^k - \lambda t\| < \|x^k\| \text{ for all } 0 < \lambda < \delta. \qquad \ldots(4)$$

PROOF : Let x^k be a minimal point of F over $X(\alpha_k)$ with $\|x^k\| = \alpha_k$, and let $t^k = x^k/\alpha_k$. Then $\|t^k\| = 1$ for all k. Since $\{t^k\}$ lies in the unit sphere which is compact, there is a subsequence that converges to some vector t in the unit sphere. Let this subsequence be one with $\alpha \in \{\alpha_k\}_{k \in \Gamma}$, where Γ is an index set. Thus we have

$$t = \lim_{k \in \Gamma} t^k = \lim_{k \in \Gamma} (x^k/\alpha_k),\ \|t\| = 1.$$

Since $x^k \in X$, $At^k = (1/\alpha_k)\, Ax^k \leqslant (b/\alpha_k)$ for all $k \in \Gamma$, which implies

$$At \leqslant 0.$$

Further, we have

$$f(\alpha_k) = \alpha_k^2\ (t^{k^T} Dt^k) + \alpha_k\, h(t^k) \text{ for all } k \in \Gamma. \qquad \ldots(5)$$

Since F is bounded below on X, it follows from (2) that $f(\alpha_k)$ is bounded below for all $k \in \Gamma$. It was seen that $f(\alpha_k)$ is also bounded above. Hence, it follows from (5) and the lower semicontinuity of h that

$$t^T Dt = 0,\ h(t) \leqslant \lim_{k \in \Gamma} h(t^k) = 0.$$

Thus, (3) is established.

Let $I = \{1, 2, \ldots, m\}$, and let $J = \{j : (At)_j < 0\}$.

Then we have

$$(At)_j < 0 \text{ for all } j \in J$$

$$(At)_j = 0 \text{ for all } j \in I \setminus J. \qquad \ldots(6)$$

Choose an index $\eta \in \Gamma$ large enough such that

$$\left.\begin{aligned} &(At^k)_j < \frac{1}{2}\,(At)_j < 0 \\ &\frac{\alpha_k}{2}\,(At)_j \leqslant b_j \end{aligned}\right\} \text{ for all } j \in J.$$

Then, for all $k \in \Gamma$ with $\alpha_k \geqslant \alpha_\eta$, we have

$$(Ax^k)_j = \alpha_k (At^k)_j < \frac{\alpha_k}{2} (At)_j \leqslant b_j \text{ for all } j \in J. \quad \ldots(7)$$

From (6) and (7) and the fact that $x^k \in X$, it follows that there is a $\delta_1 > 0$ such that

$$x^k - \lambda t \in X \text{ for all } 0 < \lambda < \delta_1 \text{ and all } k \in \Gamma. \quad \ldots(8)$$

Now, choose a fixed $k \in \Gamma$ with $\alpha_k \geqslant \alpha_\eta$ so large that $t^T t^k > 0$. Then, because of $t^T x^k > 0$ and $\|t\| = 1$, we can find a $\delta_2 > 0$ such that

$$\|x^k - \lambda t\| < \|x^k\| \text{ for all } 0 < \lambda < \delta_2. \quad \ldots(9)$$

Setting $\delta = \min(\delta_1, \delta_2)$, (4) is obtained from (8) and (9).

Lemma 2—Let y be a minimal point of F over $X(\alpha)$, $\alpha \geqslant \alpha_0$. Then the set

$$C_\alpha = \{y : y \in X(\alpha), F(y) = f(\alpha)\}$$

is compact.

PROOF : Clearly, C_α is a nonempty subset of the compact set $X(\alpha)$. Since every closed subset of a compact set is compact, we have only to show that C_α is closed. Let y^0 be a limit point of C_α, and let $\{y^i\}$ be a sequence in C_α that converges to y^0. Now for each $y^i \in \{y^i\}$, $F(y^i) = f(\alpha)$, and since F is lower semicontinuous,

$$F(y^0) \leqslant \lim_{i \to \infty} F(y^i) \leqslant f(\alpha).$$

But $\{y^i\} \subset X(\alpha)$ which is closed; and therefore $y^0 \in X(\alpha)$ and $F(y^0) \geqslant f(\alpha)$. This shows that $y^0 \in C_\alpha$, and hence C_α is closed.

Theorem 1—Let F be bounded below over X. If

$$\left.\begin{aligned} At &\leqslant 0 \\ t^T Dt &= 0 \\ h(t) &\leqslant 0 \end{aligned}\right| \Rightarrow h(t) + h(-t) = 0 \quad \ldots(10)$$

then F attains its infimum on X.

PROOF : By Lemma 2, the set $\{y : y \in X(\alpha), F(y) = f(\alpha)\}$ is nonempty and compact for any $\alpha \geqslant \alpha_0$ and the continuous function $\|y\|$ assumes its minimum on the set. Therefore, we can have, for every α, at least one vector x^α satisfying

$$x^\alpha \in X(\alpha), F(x^\alpha) = f(\alpha) \quad \ldots(11)$$

$$x^\alpha = \min(\|y\|, y \in X(\alpha), F(y) = f(\alpha)). \quad \ldots(12)$$

We now distinguish between two cases.

Case 1—$\|x^\alpha\| \neq \alpha$ for arbitrarily large values of α. This means that there exists $\bar{\alpha}$ such that $\|x^\alpha\| < \alpha$ for all $\alpha > \bar{\alpha}$. Let $\bar{f} = \inf\ (F(x) \mid x \in X)$. Since F is bounded below on $X, \bar{f}$ is finite. From (1), we then have

$$\bar{f} = \lim_{\alpha \to \infty} f(\alpha),\ f(\alpha) \geqslant \bar{f} \text{ for all } \alpha \geqslant \alpha_0 .$$

Assume that $f(\alpha) > \bar{f}$ for all $\alpha > \bar{\alpha}$. Since $f(\alpha)$ is monotone nonincreasing and tends to $\bar{f}$, we can find $\bar{\alpha} < \alpha_1 < \alpha_2$ such that $f(\alpha_1) > f(\alpha_2)$. For brevity, denote the vector x^α for $\alpha = \alpha_k$ by x^k. Because of $\bar{\alpha} < \alpha_2$, we have $\|x^2\| < \alpha_2$. Because of $f(\alpha_1) > f(\alpha_2)$, we have $\alpha_1 < \|x^2\|$. Now, choose $\alpha_3 = \|x^2\|$, and then $\bar{\alpha} < \alpha_1 < \alpha_3 < \alpha_2$. But $\bar{\alpha} < \alpha_3$ implies $\|x^3\| < \alpha_3 = \|x^2\|$, and $\alpha_3 < \alpha_2$ implies $F(x^2) \leqslant F(x^3)$. If we have $F(x^2) = F(x^3)$, then $\|x^3\| < \|x^2\|$ would contradict the definition of x^2 as given by (12), and if we have $F(x^2) < F(x^3), \|x^2\| = \alpha_3$ would contradict the definition of x^3 as given by (11). Hence, we can conclude that $\bar{f} = f(\alpha) = F(x^\alpha)$ for some α.

Case 2—$\|x^\alpha\| = \alpha$ for arbitrarily large values of α. This implies that the set of points x^α, $\alpha \geqslant \alpha_0$, is unbounded. Then we can find a sequence $\{\alpha_k\}$ with $\alpha_k \to \infty$, $\|x^k\| = \alpha_k$ for all k, where x^k corresponds to α_k. Thus we have a case to which Lemma 1 is applicable. Consequently, we get a vector t, $t \neq 0$, satisfying

$$At \leqslant 0,\ t^T Dt = 0,\ h(t) \leqslant 0. \qquad \ldots(13)$$

Since $x^k \in X$, it follows from (13) and the sublinearity of h that

$$x^k + \lambda t \in X$$

$$F(x^k + \lambda t) \leqslant F(x^k) + \lambda\,(t^T Dx^k + h(t))$$

for all k and all $\lambda \geqslant 0$. This together with the boundeness of F below on X implies

$$t^T Dx^k + h(t) \geqslant 0 \text{ for all } k. \qquad \ldots(14)$$

From (10) and (13), it follows that

$$h(t) + h(-t) = 0. \qquad \ldots(15)$$

Finally, from (13)–(15) and the sublinearity of h, it follows that

$$\begin{aligned} F(x^k - \lambda t) &\leqslant F(x^k) - \lambda\,(t^T Dx^k + h(t)) + \lambda\,(h(t) + h(-t)) \\ &\leqslant F(x^k) \end{aligned} \qquad \ldots(16)$$

for all k and all $\lambda \geqslant 0$. But by conclusion (4) of Lemma 1, we have also a scalar δ such that

$$x^k - \lambda t \in X, \|x^k - \lambda t\| < \|x^k\|$$

for some fixed k and all $0 < \lambda < \delta$. Now, if we set $y = x^k - \lambda t$ for $0 < \lambda < \delta$, then we have from (16) that

$$y \in X(\alpha_k), \|y\| < \|x^k\|$$

$$F(y) \leqslant F(x^k).$$

This contradicts the definition of x^k as given by (12). Hence, we can conclude that $\|x^\alpha\| = \alpha$ cannot hold for arbitrarily large values of α. This completes the proof of the theorem.

The existence result for quadratic functions is recovered in the following corollary.

Corollary 1—Let $F(x) = \frac{1}{2} x^T Dx + h(x)$, where $h(x)$ is any of the following functions : (i) $c^T x$, (ii) $(x^T Bx)^{1/2}$ with B positive semidefinite and (iii) $\|Sx\|_p$ $(p > 1)$ with $S \in R^{k \times n}$. If F is bounded below on X, then F attains its infimum there.

PROOF : Note that, for the functions (ii) and (iii), $h(-x) = h(x)$ and $h(x) \geqslant 0$. Since condition (10) of Theorem 1 is automatically satisfied when h assumes any of the forms (i), (ii) and (iii), the conclusion of the corollary follows from Theorem 1.

Since h is a lower semicontinuous convex function, its subdifferential $\partial h(x)$ is nonempty, compact and convex (Rockafellar[6]). Consider the complementarity problem (CP) : Find $x \in R^n$ such that

$$x \geqslant 0, \; z \geqslant 0$$

$$z \in Dx + \partial h(x), \; x^T z = 0.$$

Generalized complementarity problem of this form were studied by Saigal[7] and others, and may be of interest in non-differentiable programming.

We give the following existence theorem for (CP).

Theorem 2—Let F be bounded below on $x \geqslant 0$. If

$$\left.\begin{array}{ll} t & \geqslant 0 \\ t^T Dt & = 0 \\ h(t) & \leqslant 0 \end{array}\right| \Rightarrow h(t) + h(-t) = 0$$

then (CP) has a solution.

PROOF : By Theorem 1, F attains its infimum on $x \geqslant 0$. Let x^0 be a minimal point of F. As a necessary condition of optimality (Parida *et al.*[4]) we have a vector $y^0 \in \partial h\,(x^0)$ such that

$$x^0 \geqslant 0,\ Dx^0 + y^0 \geqslant 0$$

$$y^0 \in \partial h\,(x^0),\ \ x^{0^T}\,(Dx^0 + y^0) = 0.$$

This shows that x^0 is a solution to (CP).

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