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SEMINORMABILITY OF CERTAIN RING TOPOLOGIES ON DEDEKIND DOMAINS

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Criteria are given for seminormability of certain ring topologies on Dedekind domains. This generalizes a theorem of Cohen [*Canad. J. Math.* 33 (1981), 571] for normability of certain ring topologies on Dedekind domains and simplifies the criteria given by Warner [*Proc. Am. Math. Soc.* 33 (1972), 423] for ring topologies defined by natural pseudovaluations on Dedekind domains.

Let T be a ring topology on a ring R, i. e., T is a topology on R making $(x, y) \longrightarrow x - y$ and $(x, y) \longrightarrow xy$ continuous from $R \times R$ to R. A subset A of R is 'bounded' if given any neighbourhood U of zero, there exists a neighbourhood V of zero such that $AV \subseteq U$ and $VA \subseteq U$. T is 'locally bounded' if there exists a bounded neighbourhood of zero for T. An element x in R is called a 'topological nilpotent' if $\lim_{n \to \infty} x^n = 0$.

We recall that a real valued function N defined on a ring R is called a 'seminorm' if for all x, y in R, $N(x) \geqslant 0$, $N(0) = 0$, $N(-x) = N(x)$, $N(x+y) \leqslant N(x) + N(y)$ and $N(xy) \leqslant N(x)\, N(y)$. A seminorm N on R becomes a norm on R if $N(x) = 0$ implies $x = 0$. A seminorm (norm) N on R becomes an 'absolute semivalue' absolute value) if $N(xy) = N(x)\, N(y)$ for all x, y in R. A seminorm (norm, absolute semivalue, absolute value) N on R is called non-archimedean if $N(x + y) \leqslant \text{Max}\,\{N(x), N(y)\}$ for all x, y in R. A seminorm or an absolute semivalue (norm or absolute value) N on a ring R defines a locally bounded (Hausdorff, locally bounded) ring topology T_N on R. Two seminorms (norms,...) on a ring R are said to be equivalent if they define the same topologies on R. A function v defined on a ring R is a 'natural pseudovaluation' if the range of v is contained in the natural numbers together with 0 and $+\infty$ such that for all $x,y \in A$, $v(-x) = v(x)$; $v(x+y) \geqslant \text{Min}\,\{v(x), v(y)\}$; $v(xy) \geqslant v(x) + v(y)$ and $v(0) = +\infty$. Clearly the exponential of a natural pseudovaluation to a base less than 1 is a non-archimedean seminorm N satisfying $N(x) \leqslant 1$ for all $x \in A$. N is a norm if v satisfies $v(x) = +\infty$ iff $x = 0$. A ring D is said to be a Dedekind domain if it is an integral domain and if every ideal in D is product of prime ideals. It is well known that in a Dedekind domain the factorization of ideals into prime ideals is unique (Zariski and Samuel[4], Chap. V. Corollary of Theorem 10].

Let D be a Dedekind domain that is not a field. Let F be the quotient field of D and P be the set of non-zero prime ideals of D. Every non-zero fractional ideal A of F has the unique factorization.

$$A = \pi_{p \in P} \; p^{n_p(A)} \text{ where } n_p(A) \in Z \text{ and } n_p(A) = 0$$

for all but finitely many $p \in P$. If A and B are two non-zero fractinol ideals of F, then $A \subseteq B$ if and only if $n_p(A) \geqslant n_p(B)$ for all $p \in P$; also $n_p(A + B) =$ Min $\{n_p(A), n_p(B)\}$ for all $p \in P$ [Zariski and Samuel[4], Chap.V, Theorem 11]. For each $p \in P$, let N_p be the absolute value on F defined by $N_p(0) = 0$ and for $x \neq 0$ in $F, N_p(x) = 2^{-n_p(Dx)}$. Then N_p restricted to D defines a non-discrete, Hausdorff, locally bounded ring topology on D which for the sake of brevity will be denoted by T_p. If $p \in P$ and n is a fixed positive integer, the function N'_{np} defined on D by

$$N'_{np}(x) = 0 \text{ if } x \in p^n \text{ and } 1 \text{ otherwise,}$$

is a seminorm. We denote by T'_{np}, the topology defined by this seminorm. Note that the singleton set $\{p^n\}$ is a fundamental system of neighbourhoods of zero for T'_{np}.

The following theorem gives sufficient conditions for seminormability of ring topologies on a Dedekind domain. This refines Theorem 1 of Cohen[1] in two ways. First the set J and S give a somewhat more explicit characterization of the topology T. Secondly it extends the result to non-Hausdorff topologies.

Theorem 1—Let T be a ring topology on a Dedekind domain D. Suppose that the open ideals of D form a fundamental system of neighbourhoods of zero for T (therefore T is locally bounded) and that there exists a non-zero topological nilpotent for T. Let

$$J = \{ d \in D / \text{ there exists a non-zero } t \text{ in } D \text{ with } d^n t \longrightarrow 0\}$$

$$S = \{ t \in D / d^n t \longrightarrow 0 \text{ for each } d \in J\}.$$

Then

(1) J is a non-zero ideal of D and if $J \neq D$,

$J = p_1 p_2 \ldots p_r$, where $p_1, p_2, \ldots, p_r$ are distinct prime ideals of D.

(2) S is a non-zero ideal of D and $(S, J) = 1$, where (S, J) stands for the greatest common divisor of S and J. For $S \neq D$, let $S = q_1^{e_1} q_2^{e_1} \ldots q_s^{e_s}$, where $q_1, q_2, \ldots, q_s$ are distinct prime ideals of D and $e_1, e_2, \ldots, e_s$ are positive integers

3) Either the topology T on D is the trivial topology or

$$T = \sup \left[\sup_{1 \leqslant i < r} T_{p_i}, \sup_{1 \leqslant j \leqslant s} T'_{e_j} {}^{q_j} \right].$$

PROOF : (1) Let d_1, d_2 be two non-zero elements of J and t_1, t_2 be non-zero elements of D for which $d_1^n t_1 \longrightarrow 0$ and $d_2^n t_2 \longrightarrow 0$. Let U be a T-open ideal and k be a positive integer such that $d_1^k t_1, d_2^k t_2 \in U$. Then for all $n \geqslant 2k$ and $a_1, a_2 \in D$, we have

$$(a_1 d_1 + a_2 d_2)^n \; t_1 t_2 = (a_1 d_1 + a_2 d_2)^{n-2k}$$

$$\left[\sum_{i=0}^{2k} \binom{2k}{i} a_1^i d_1^i a_2^{2k-i} d_2^{2k-i} t_1 t_2 \right]$$

$$= (a_1 d_1 + a_2 d_2)^{n-2k} \left[d_2^k t_2 \sum_{i=0}^{k} \binom{2k}{i} a_1^i d_1^i d_2^{2k-i} d_2^{k-i} t_1 \right.$$

$$\left. + d_1^k t_1 \sum_{i=k+1}^{2k} \binom{2k}{i} a_1^i d_1^{i-k} a_2^{2k-i} d_2^{2k-i} t_2 \right]$$

$$\in D[UD + UD] \subseteq U.$$

Therefore $a_1 d_1 + a_2 d_2 \in J$.

As D has a non-zero topological nilpotent, say u, we have $u^n.1 \longrightarrow 0$ and hence J is a non-zero ideal of D. If $J \neq D$, there exist non-zero prime ideal $p_1, p_2 \ldots, p_r$ and positive integers $\alpha_1, \alpha_2, \ldots, \alpha_r$ satisfying

$$J = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}.$$

Let $\alpha = \text{Max}\ \{\alpha_1\ \alpha_2, \ldots, \alpha_r\}$ and $x_i \in p_i$ for each $i = 1, 2, \ldots, r$. Then $(x_1\ x_2, \ldots, x_r)^\alpha \in J$, so there exists a non-zero element t in D such that $(x_1\ x_2 \ldots x_e)^{\alpha n}\ t \longrightarrow 0$ and hence $x_1\ x_2 \ldots x_r \in J$. As J is an ideal of D, $p_1\ p_2 \ldots p_r \subseteq J = p_1^{\alpha_1}\ p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, so $\alpha_i = 1$ for $i = 1, 2, \ldots, r$.

(2) Clearly $0 \in S$ and if $t_1, t_2 \in S$ then for each $d \in J$, $d^n\,(t_1 \pm t_2) = d^n t_1 \pm d^n t_2 \longrightarrow 0$ and hence $t_1 \pm t_2 \in S$. Therefore S is an ideal of D. We now prove S to be non-zero. As J is a non-zero ideal of D there exist h_1, h_2 in J such that $J = Dh_1 + Dh_2$ [Zariski and Samuel[4], Chap. V,C orollary 2 of Theorem 16]. Let t_1, t_2 be non-zero elements of D for which $h_1^n\ t_1 \longrightarrow 0$ and $h_2^n\ t_2 \longrightarrow 0$. Every elenent h of J can be expressed as $h = a_1\ h_1 + a_2\ h_2$ for $a_1, a_2 \in D$. From the proof of (1) we get $h^n t_1 t_2 \longrightarrow 0$. Therefore $t_1\ t_2$ is a non zero element of S.

If $S \neq D$, S can be expressed as $S = q_1^{e_1}\ q_2^{e_2} \ldots q^e{}_s$ where $q_1, q_2, \ldots, q_s$ are distinct non-zero prime ideals of D and $e_1, e_2, \ldots e_s$ are positive integers. We now prove that $(S, J) = 1$. Let t be a non-zero element of S. Let $Dt = AB$ where A is the product of powers of those prime ideals which also appear in J and B the product of powers of those prime ideals which do not appear in J. There exists a positive integer α such that J^α is divisible by A. By Chinese Remainder Theorem (Zariski and Samuel[4], Chap. V. Theorem 17] we can choose an element t^* in D such that the ideal Dt^* is divisible by the ideal B but coprime to the ideal J. As $t \in S$, so $d^n t \longrightarrow 0$ for each $d \in J$. Let $d \in J$, $d \neq 0$, be fixed. Then the ideal $Dd^\alpha t^*$ is divisible by the ideal $J^\alpha B$ which implies that the ideal $Dd^\alpha t^*$ is divisible by the ideal AB and hence $d^\alpha\ t^* = t\ t'$ for some $t' \neq 0$ in D. Now $d^n t^* = d^{n-\alpha}\ d^\alpha t^* = (d^{n-\alpha} t)\ t' \longrightarrow 0$ as $n \longrightarrow \infty$. Therefore $t^* \in S$ and hence the ideal Dt^* is contained in the ideal S. There exists an ideal C such that $Dt^* = CS$ [Larsen and McCarthy[2], Theorem 6.20 (8)]. As the ideal Dt^* is coprime to the ideal J, S is coprime to J.

(3) First we note that if $J = D$ and $S = D$, each element of D is a topological nilpotent and hence the topology T on D is the trivial topology. Assume, then, that one of J and S is not equal to D. We now prove that if we W is a T-open ideal of D, then

$$W = p_1^{\alpha_1}\ p_2^{\alpha_2}\ \ldots\ p_r^{\alpha_r}\ q_1^{\beta_1}\ q_2^{\beta_2}\ \ldots\ q_s^{\beta_s}$$

where $\alpha_i \geqslant 0$ and $0 \leqslant \beta_j \leqslant e_j$ for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$.

As J is an ideal of D, we can find h_1, h_2 in J such that $J = Dh_1 + Dh_2$. Let t be a non-zero element of S. Then $h_1^n\ t \longrightarrow 0$ and $h_2^n\ t \longrightarrow 0$. Also

$$p_1\ p_2 \ldots p_r = J = Dh_1 + Dh_2 = \underset{p \in P}{\pi}\, p^{\text{Min}\ \{n_p(Dh_1),\, n_p(Dh_2)\}},$$

so for each $p \in P$, $p \neq p_1, p_2, \ldots p_r$, either $n_p(Dh_1) = 0$ or $n_p(Dh_2) = 0$. For each positive integer n,

$$Dh_1^n t = \prod_{p \in P} p^{n_p(Dh_1^n t)} \quad \text{and} \quad Dh_2^n t = \prod_{p \in P} p^{n_p(Dh_2^n t)}.$$

Let k be positive integer such that $h_1^k t$, $h_2^k t \in W$. Then $Dh_1^k t \in W$ and $Dh_2^k t \in W$.

Therefore for $p \in P$, $p \neq p_1, p_2, \ldots, p_r$,

$$n_p(W) \leqslant kn_p(Dh_1) + n_p(Dt)$$

and

$$n_p(W) \leqslant kn_p(Dh_2) + n_p(Dt)$$

so for all $p \in P$, $p \neq p_1, p_2, \ldots, p_r$, $n_p(W) \leqslant n_p(Dt)$. As t is is an arbitrary element of S and $(J, S) = 1$, we get $n_p(W) = 0$. Also from the above two inequalities, we get $n_{q_j}(W) \leqslant e_j$ for $j = 1, 2, \ldots, s$.

Hence

$$W = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r} q_1^{\beta_1} q_2^{\beta_2} \ldots q_s^{\beta_s}$$

where $\alpha_i \geqslant 0$ for $i = 1, 2, \ldots, r$ and $0 \leqslant \beta_j \leqslant e_j$ for $j = 1, 2, \ldots, s$.

Next we prove that given any positive integer α_i $1 \leqslant i \leqslant r$, there exists an open ideal W_i with $n_{p_i}(W_i) \geqslant \alpha_i$. Suppose it is not true some i. Without loss of generality, assume $i=1$. Let $\alpha = \text{Sup}\,\{n_{p_1}(W) : W \text{ is a } T \text{ open}\} < \infty$. Let $x_i \in p_i$ for $i = 1, 2, \ldots r$, $x = x_1 x_2 \ldots x_r$ and $y = x_2 x_3 \ldots x_r$. There exists a non-zero element t in D such that $x^n t \to 0$. Let W be any T-open ideal. Therefore

$$W = p_1^{\alpha_1} \; p_2^{\alpha_2} \ldots p_r^{\alpha_r} \; q_1^{\beta_1} \ldots q_s^{\beta_s}$$

where $0 \leqslant \alpha_1 \leqslant \alpha$, $\alpha_i > 0$ for $i = 2, 3, \ldots, r$ and $0 \leqslant \beta_j \leqslant e_j$ for $j = 1, 2, \ldots, s$. There exists a positive integer k such that for all $n \geqslant k$, $x^n t \in W$. One can verify that $y^n(x_1^\alpha t) \in W$ for all $n \geqslant k$. Therefore $y \in J$. Since J is an ideal, $p_2 p_3 \ldots p_r \subseteq J = p_1 p_2 \ldots p_r$ which is a contradiction. Similarly we can prove that given any integer β_j, $0 \leqslant \beta_j \leqslant e_j$ for $j = 1, 2, \ldots, s$, there exists a T-open ideal U_j such that $n_{q_j}(U_j) = \beta_j$.

Let $\left(\bigcap_{1 \leqslant i \leqslant r} p_i^{\alpha_i'} \right) \cap \left(\bigcap_{1 < i \leqslant s} q_j^{\beta_j'} \right)$ be a neighbourhood of zero for $\operatorname{Sup}\left\{\sup_{1\leqslant i<r} T_{p_i}, \sup_{1<j\leqslant s} T'_{e_{j}q_j}\right\}$ where $\alpha_i \geqslant 1$ for $1 \leqslant i \leqslant r$.

We can find T-open ideals W_i and U_i such that $n_{p_i}(W_i) \geqslant \alpha_i$ and $n_{q_j}(U_j) = e_j$ for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$.

So the T-open ideal $(\bigcap_{1<i<r} W_i) \cap (\bigcap_{1\leqslant j<s} U_j)$ is contained in $(\bigcap_{1\leqslant i\leqslant r} p_i^{\alpha_i}) \cap (\bigcap_{1\leqslant j\leqslant s} q_j^{e_j})$. Therefore, the supremum topology is weaker than T. Finally each T-open ideal is of the form $(\bigcap_{1<i\leqslant r} p_i^{\alpha_i}) \cap (\bigcap_{1<j<s} q_j^{\beta_j})$ where $\alpha_i \geqslant 0$ for $i = 1, 2, \ldots, r$ and $0 \leqslant \beta_j \leqslant e_j$ for $j = 1, 2, \ldots, s$, so T is weaker than the supremum topology.

The following theorem gives necessary and sufficient conditions for a ring topology on Dedekind domain D to be defined by a natural pseudovaluation, and hence by a non-archimedean seminorm N satisfying $N(x) \leqslant 1$ for all $x \in D$. This simplifies the criteria given by Warner[8].

Theorem 2—The ring topology T on a Dedekind domain D is defined by a natural pseudovaluation v if and only if the following two conditions hold :

(1) The open ideals of D from a fundamental system of neighbourhoods of zero.

(2) Either the topology T is discrete or there exists a non-zero topological nilpotent element for T.

PROOF : Necessity : Let $A_n = \{x \in D : v(x) > n\}$. Then $\{A_n\}_{n\geqslant 1}$ is a fundamental system of neighbourhoods of zero for T. If $v(x) = 0$ for all $x \neq 0$ in D, then the topology T defined by v is discrete, otherwise each non-zero x for which $v(x) \neq 0$ is a topolgical nilpotent element for T in D.

Sufficiency : T is defined by the seminorm N given by

$$N(x) = \operatorname{Sup}\left\{\sup_{1\leqslant i\leqslant r} N_{p_i}(x), \sup_{1\leqslant j\leqslant s} N'_{e_j q_j}(x)\right\} \text{ for all } x \in D.$$

which is always a negative integral power of 2, i. e., $N(0) = 0$ and $N(x) = 2^{-v(x)}$ for all $x \in D$. Then $v(x)$ is a natural pseudovaluation on D defining the given topology T on D.

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