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LOCALLY SEMI-CONNECTEDNESS IN TOPOLOGICAL SPACES

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In this paper the idea of local semi-connectedness in a topological space is introduced and obtain some of its basic properties.

1. INTRODUCTION

After the introduction of the concept of semi-open sets by Norman Levine[5], various authors have turned their attentions to this concept and it becomes the primary aim of many mathematicians to examine and explore how far the basic concepts and theorems remain true if one replaces open set by semi-open set. The concept of semi-connectedness in topological spaces has been introduced by Das[2] and with the help of this concept we here introduce the idea of local semi-connectedness in a topological space and obtain some basic properties.

2. KNOWN DEFINITIONS

Definition 2.1 (Levine[5])—A set A in a topological space (X, T) will be said to be semi-open if and only if there exists an open set 0 such that $0 \subset A \subset \bar{0}$, where $\bar{0}$ denotes the closure of 0 in X.

Definition 2.2 (Biswas[1])—A set A in a topological space (X, T) is said to be semi-closed if and only if there exists a closed set F such that $\overset{\circ}{F} \subset A \subset F$, where $\overset{\circ}{F}$ denotes the interior of F in X.

Definition 2.3 (Biswas[1])—The intersection of all semi-closed sets containing a subset A of a topological space (X, T) is said to be the semi-closure of A and is denoted by A.

The present work is dedicated to the memory of Prof. Norman Levine whose sudden demise left a void in the field of Topology. The second author is inspired in research by the works of Prof. Levine.

Definition 2.4 (Das[2])—Two non-null subsets A, B of a topological space (X, T) are said to be semi-separated if and only if $A \cap \mathbf{B} = \mathbf{A} \cap B = \phi$, where ϕ denotes the null set.

Definition 2.5 (Das[2])—In a topological space (X, T), a set which cannot be expressed as the union of two semi-separated sets is said to be a semi-connected set. The topological space (X, T) is said to be semi-connected if and only if X is semi-connected.

Definition 2.6 (Levine[5])—A mapping $f: (X, T) \to (X^*, T^*)$ is said to be semi-continuous if for each open set V^* in (X^*, T^*), $f^{-1}(V^*)$ is semi-open in (X, T).

3. Known Theorems

Theorem 3.1 (Noiri[6])—Let (X, T) and (X^*, T^*) be two topological spaces. If $f: X \to X^*$ is an open and semi-continuous mapping, then the inverse image $f^{-1}(B)$ of each semi-open set B in (X^*, T^*) is semi-open in (X, T).

Theorem 3.2 (Dorsett[3])—If f is a semi-continuous mapping from a semi-connected space (X, T) onto (X^*, T^*) then (X^*, T^*) is semi-connected.

Theorem 3.3 (Dorsett[3])—Let (X, T) be a topological space and A be open. Then A is semi-connected if and only if $(A, T/A)$ is semi-connected, where T/A is the induced topology of T into A.

Theorem 3.4 (Th. 1.5, Remark 1.5 of Grossley and Hildebrand[4])—If A is a subset of a topological space (X, T), then $A \subsetneq \mathbf{A}$ and $\mathbf{A}$ is semi-closed.

Theorem 3.5 (Th. 1.4 of Crossley and Hildebrand[5])—A subset A of a topological space (X, T) is semi-closed if and only if $\mathbf{A} = A$.

4. New Definitions and Lemmas

Definition 4.1—A topological space (X, T) is called locally semi-connected at $x \in X$ if and only if for every semi-open set U containing x, there exists a semi-connected open set C such that $x \in C \subseteq U$. (X, T) is called locally semi-connected if and only if it is locally semi-connected at every point of X.

Remark 4.1 : If a topological space (X, T) is locally semi-connected, then it is locally connected but the converse is not true as shown by the following example :

Example 4.1—We consider the topological space (X, T) where $X = (a, b, c)$ and $T = (X, \phi, (a), (a, b))$.

Here, $C(T)$ = Class of all closed sets = $(\phi, X, (b, c), (c))$, $SO(T)$ = Class of all semi-open sets = $(X, \phi, (a), (a, b), (a, c))$ and $SC(T)$ = Class of all semi-closed sets = $(\phi, X, (b, c), (c), (b))$.

It is verified that X is locally connected. But we show that X is not locally semi-connected.

Here (a, c) is a semi-open set containing c, but there is no open subset of (a, c) containing c and so X is not locally semi-connected at c. Therefore X is not locally semi-connected.

Remark 4.2 : Local semi-connectedness does not imply semi-connectedness as shown by the following example :

Example 4.2—We consider the topological space (X, T), where $X = (a, b, c)$ and $T = (X, \phi, (a), (a, b), (a,c), (c))$.

With the notations used in Example 4.1, we have $C(T) = (\phi, X, (b, c), (c), (b), (a, b))$, $SO(T) = (X, \phi, (a), (a, b), (a, c), (c))$ and $SC(T) = (\phi, X, (b, c), (c), (b), (a,b))$.

The semi-open sets containing a are (a), (a, b), (a, c) and X. Clearly the set (a) is semi-connected and open. Therefore X is locally semi-connected at a.

The semi-open sets containing b are (a, b) and X. We show that (a, b) is semi-connected. Let $A = (a)$, $B = (b)$. Then $\mathbf{A} = (a, b)$ and so $B \cap \mathbf{A} \neq \phi$. Therefore (a, b) is semi-connected and open and so X is locally semi-connected at b.

The semi-open sets containing c are (c), (a, c) and X. Clearly the set (c) is semi-connected and open and so X is locally semi-connected at c. Therefore X is locally semi-connected.

Now we show that X is not semi-connected. Let $A = (a, b)$ and $B = (c)$. Then $\mathbf{A} = (a, b)$ and $\mathbf{B} = (c)$, $A \cap \mathbf{B} = \phi$ and $B \cap \mathbf{A} = \phi$ and so A and B are two semi-separated sets. Hence X can be expressed as the union of two semi-separated sets and so X is not semi-connected.

Remark 4.3 : Semi-connectedness does not imply local semi-connectedness as shown by the following example :

Example 4.3—We consider the topological space of Example 4.1. Here we show that X is semi-connected. We first choose $A = (a)$, $B = (b, c)$. Then $\mathbf{A} = X$ and so $B \cap \mathbf{A} \neq \phi$.

We next choose $A = (b)$, $B = (a, c)$. Then $\mathbf{B} = X$ and so $A \cap \mathbf{B} \neq \phi$.

Lastly we choose $A = (c)$, $B = (a, b)$. Then $\mathbf{B} = X$ and so $A \cap \mathbf{B} \neq \phi$.

Thus we see that X cannot be expressed as the union of two semi-separated sets and hence X is semi-connected.

But in Example 4.1 we have seen that X is not locally semi-connected.

Lemma 4.1—If A and B are two subsets of a topological space (X, T) and $A \subset B$, then $\mathbf{A} \subset \mathbf{B}$.

PROOF : The proof is omitted.

Lemma 4.2.—If A is semi-connected and $A \subset C \cup D$ where C and D are semi-separated, then either $A \subset C$ or $A \subset D$.

PROOF : We write $A = (A \cap C) \cup (A \cap D)$. Then by Lemma 4.1, we have $(A \cap C) \cap (\mathbf{A} \cap \mathbf{D}) \subset C \cap \mathbf{D}$.

Now since C and D are semi-separated, $C \cap \mathbf{D} = \phi$ and so $(A \cap C) \cap (\mathbf{A} \cap \mathbf{D}) = \phi$.

Similarly, $(A \cap D) \cap (\mathbf{A} \cap \mathbf{C}) = \phi$.

So if both $A \cap C \neq \phi$ and $A \cap D \neq \phi$, then A is not semi-connected. This contradiction proves that either $A \cap C = \phi$ or $A \cap D = \phi$, which again implies that either $A \subset C$ or $A \subset D$.

This proves the lemma.

Lemma 4.3—The union E of any family (C_λ) of semi-connected sets having a non-empty intersection is a semi-connected set.

PROOF : If E is not semi-connected, we can write $E = A \cup B$, where A and B are semi-separated sets. By hypothesis, we may choose a point $x \in \bigcap_\lambda C_\lambda$. The point x must belong to either A or B and without loss of generality let us suppose that $x \in A$. Since x belongs to C_λ for every λ, $C_\lambda \cap A \neq \phi$ for every λ. By the Lemma 4.2, however each C_λ must be either a subset of A or a subset of B. Since A and B are disjoint sets we must have $C_\lambda \subset A$ for all λ and so $E \subset A$. From this we obtain the contradiction that $B = \phi$. This proves the lemma.

Lemma 4.4—If C is semi-connected and $C \subset E \subset \mathbf{C}$, then E is semi-connected.

PROOF : If E is not semi-connected, then we can write $E = A \cup B$ where $A \neq \phi$, $B \neq \phi$, $A \cap \mathbf{B} = \phi = \mathbf{A} \cap B$. By the Lemma 4.2, we must have $C \subset A$ or $C \subset B$. Without loss of generality let us suppose that $C \subset A$. From this it follows that $\mathbf{C} \subset \mathbf{A}$ and hence $\mathbf{C} \cap B \subset \mathbf{A} \cap B = \phi$. On the other hand, $B \subset E \subset \mathbf{C}$ and so $\mathbf{C} \cap B = B$. Thus we must have $B = \phi$, which is a contradiction. This proves the lemma.

Lemma 4.5—Let $f : (X, T) \to (X^*, T^*)$ be open, semi-continuous and $A \subset X$ be open. Then if A is semi-connected, $f(A)$ is also semi-connected.

PROOF : Since A is semi-connected and open in (X, T), then $(A, T/A)$ is also semi-connected by the Theorem 3.3.

Now, $f/A : (A, T/A) \to (f(A), T^*/f(A))$ is onto and semi-continuous and so by Theorem 3.2, $f(A)$ is semi-connected in $(f(A), T^*/f(A))$.

Now since f is open, $f(A)$ is open in (X^*, T^*) and so by Theorem 3.3, $f(A)$ is semi-connected in $(X^* T^*)$.

This proves the lemma.

According to Definition 2.7 of Das[2] semi-components of a topological space (X, T) are maximal semi-connected subsets of (X, T). If an element x is a member of some semi-component, then that semi-component will be denoted by S. C. (x). We give below a formal definition of S. C (x).

Definition 4.2—Let (X, T) be a topological space and $x \in X$. The semi-component of x, denoted by S. C (x), is the union of all semi-connected subsets of X containing x. The sets like S. C (x) are called semi-components of X.

Further if $E \subset X$ and if $x \in E$, then the union of all semi-connected sets containing x and contained in E is called the semi-component of E corresponding to x. By the term that C is a semi-component of E, we mean that C is a semi-component of E corresponding to some point of E.

As the union of any family of semi-connected sets having a non-empty intersection is a semi-connected set by Lemma 4.3, it follows that S. C (x) is semi-connected.

5. Basic Properties

Theorem 5.1—In a topological space (X, T),

(i) each semi-component S. C. (x) is a maximal semi-connected set in X,

(ii) the set of all distinct semi-components of points of X form a partition of X and

(iii) each S. C (x) is semi-closed in X.

PROOF : (i) follows from the Definition 4.2.

(ii) Let S.C $(x)$and S.C (x') be two semi-components of distinct points x and x' in X. If S.C $(x) \cap$ S.C $(x') \neq \phi$, then by Lemma 4 3, S.C $(x) \cup$ S.C (x') is semi-connected. But S.C $(x) \subset$ S.C $(x) \cup$ S.C (x') which contradicts the maximality of S.C (x).

Let x be any point in X. So $x \in$ S.C (x). Now $\bigcup_{x \in X} (x) \subset \bigcup_{x \in X}$ S.C (x).

This implies that $X \subset \bigcup_{x \in X}$ S.C $(x) \subset X$. Therefore $\bigcup_{x \in X}$ S.C $(x) = X$.

(iii) Let x be any point in X. Then S.C (x) is a semi-connected set containing x by Lemma 4.4. But S.C (x) is the maximal semi-connected set containing x. So S.C (x) $\subset$ S.C (x). Hence S.C (x) is semi-closed by Theorem 3.5.

Theorem 5.2—A topological space (X, T) is locally semi-connected if and only if the semi-components of semi-open set are open sets.

PROOF : Suppose that (X, T) is locally semi-connected. Let $G \subset X$ be semi-open and C be a semi-component of G. If $y \in C$, then because $y \in G$, there is a semi-connected open set U such that $y \in U \subset G$. Since C is the semi-component of y and U is semi-connected, we have $y \in U \subset C$. This shows that C is open.

Conversely, let $x \in X$ be arbitrary and let G be a semi-open set containing x. Let C be the semi-component of G such that $x \in C$. Now C is a semi-connected open set such that $x \in C \subset G$.

This proves the theorem.

Theorem—5.3 Let (X, T) and (X^*, T^*) be two topological spaces and $f: (X, T) \rightarrow (X^*, T^*)$ be a mapping which is semi-continuous, open and onto. Then if X is locally semi-connected, X^* is locally semi-connected.

PROOF : Let U be any semi-open subset of X^* and C be any semi-component of U. As f is open and semi-continuous, $f^{-1}(U)$ is semi-open by the Theorem 3.1. Let A be any semi-component of $f^{-1}(U)$. Since X is locally semi-connected and since $f^{-1}(U)$ is semi-open, A is open by the Theorem 5.2. Also by Lemma 4.5, $f(A)$ is a semi-connected subset of X^* and since C is a semi-component of U, it therefore follows that either $f(A) \subset C$ or $f(A) \cap C = \phi$. Thus $f^{-1}(C)$ is the union of collection of semi-components of $f^{-1}(U)$ and so $f^{-1}(C)$ is open. As f is open and onto, $C = ff^{-1}(C)$ is open in X^*. Thus any semi-component of semi-open set in X^* is open in X^* and hence by Theorem 5.2, X^* is locally semi-connected.

Theorem 5.4— A topological space (X, T) is locally semi-connected if and only if, given any point $x \in X$ and a semi-open set U containing x, there is an open set C containing x such that C is contained in a single semi-component of U.

PROOF : Let X be locally semi-connected, $x \in X$ and U be a semi-opne set containing x. Let A be a semi-component of U that contains x. Since X is locally semi-connected and U is semi-open, there is a semi-connected open set C such that $x \in C \subset U$. By Theorem 5.1, A is the maximal semi-connected set containing x and so $x \in C \subset A \subset U$. Since semi-components are disjoint sets, it follows that C is not contained in any other semi-component of U.

Conversely, we suppose that given any point $x \in X$ and any semi-open set U containing x, there is an open set C containing x which is contained in a single semi-component F of U. Then $x \in C \subset F \subset U$. Let $y \in F$, then $y \in U$. Thus there is an open set 0 such that $y \in 0$ and 0 is contained in a single semi-componet of U. As the semi-components are disjoint sets and $y \in F$, $y \in 0 \subset F$. Thus F is open.

Thus for every $x \in X$ and for every semi-open set U containing x, there is a semi-connected open set F such that $x \in F \subset U$. Thus (X, T) is locally semi-connected at x. Since $x \in X$ is arbitrary, (X, T) is locally semi-connected.

This proves the theorem.

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