ON SEMI SEPARATION PROPERTIES

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This paper considers semi separation axioms for topological spaces, especially their relationship to the separation properties of the associated α-topologies. The semi-$T_2$ property is characterized in terms of regular semi-open sets, and shown to be a semi-regular property. It is also established that semi-regular properties are shared by a topological space and its associated α-space.

1. INTRODUCTION

Let $B$ be a subset of a topological space $(X, \mathcal{F})$. We denote the closure of $B$ and the interior of $B$ with respect of $\mathcal{F}$ by $\overline{B}$ and $\text{int } B$ respectively, although we may suppress the $\mathcal{F}$ when there is no possibility of confusion. A subset $B$ of $(X, \mathcal{F})$ is said to be regular open if $B = \text{int } (\overline{B})$, semi-open if $B \subseteq \overline{\text{int } B}$, and semi-closed if $\text{int } (\overline{B}) \subseteq B$. The collection of all regular open [semi-open, semi-closed respectively] subsets of $(X, \mathcal{F})$ is denoted by $RO(X, \mathcal{F}), SO(X, \mathcal{F}), SC(X, \mathcal{F})$ respectively. Njastad\textsuperscript{10} defined $B$ to be an α-set in $(X, \mathcal{F})$ if $B \subseteq \text{int } (\overline{\text{int } B})$, and showed that the family $\mathcal{F}^*$ of all α-sets in $(X, \mathcal{F})$ is a topology on $X$ larger than $\mathcal{F}$. He used the term β-set for semi-open set, and showed that $SO(X, \mathcal{F}) = SO(X, \mathcal{F}^*)$. The topology on $X$ having $RO(X, \mathcal{F})$ as a base is called the semi-regularization of $\mathcal{F}$, and is denoted by $\mathcal{F}_s$.

In recent years there has been a considerable number of papers\textsuperscript{4,5,7,8,11} considering separation properties, essentially defined by replacing open sets by semi-open sets. In this paper, we relate these semi-separation properties of $(X, \mathcal{F})$ to the separation properties of $(X, \mathcal{F}^*)$. We show, for example, that $(X, \mathcal{F})$ is semi-$T_6$ if
and only if \((X, \mathcal{F}^*)\) is \(T_0\), and that \((X, \mathcal{F}^*)\) is \(T_1\) implies \((X, \mathcal{F})\) is semi-\(T_1\) but not conversely.

If \(B\) is a subset of \((X, \mathcal{F})\) then the semi-closure\(^3\) of \(B\), denoted by \(\text{scl } B\), is the intersection of all semi-closed subsets of \(X\) containing \(B\). For any set \(B\) in \((X, \mathcal{F})\) we have \(\text{scl } B \subseteq \text{cl } B\). Our first result expresses the semi-closure of a set in terms of interior and closure.

**Lemma—1** Let \(A\) be a subset of \((X, \mathcal{F})\). Then \(\text{scl } A = A \cup \text{int } (\text{cl } A)\).

**Proof**: Since \(\text{scl } A\) is semi-closed, we have \(\text{int } (\text{cl } (\text{scl } A)) \subseteq \text{scl } A\). Therefore, \(\text{int } (\text{cl } A) \subseteq \text{scl } A\), and hence \(A \cup \text{int } (\text{cl } A) \subseteq \text{scl } A\). To establish the opposite inclusion we observe that \(\text{int } (\text{cl } (A \cup \text{int } (\text{cl } A))) = (\text{cl } A) \cup \text{int } (\text{cl } (\text{cl } A))\) 
\(\subseteq (\text{cl } A) \cup \text{int } (\text{cl } (\text{cl } A)) = (\text{cl } A) \cup \text{int } (\text{cl } A) = \text{cl } A\). Thus \(\text{int } (\text{cl } (A \cup \text{int } (\text{cl } A))) \subseteq \text{int } (\text{cl } A) \subseteq A \cup \text{int } (\text{cl } A)\). Hence \(A \cup \text{int } (\text{cl } A)\) is semi-closed, and so \(\text{scl } A \subseteq A \cup \text{int } (\text{cl } A)\).

Maheshwari and Tapi\(^a\) have introduced the notion of feebly open sets. The subset \(B\) of \((X, \mathcal{F})\) is called feebly open if there is an open set \(U\) such that \(U \subset B \subset \text{scl } U\). A set is feebly closed if its complement is feebly open, and the intersection of all feebly closed sets containing \(B\) is the feebly closure of \(B\), denoted by \(\text{fcl } B\). Our next result shows that these notions coincide with the existing concepts of \(\alpha\)-closed set, and \(\mathcal{F}^*\) closure, respectively.

**Proposition 1**—Let \(A\) be a subset of \((X, \mathcal{F})\). Then \(A\) is feebly open if and only if \(A \in \mathcal{F}^*\).

**Proof**: If \(A\) is feebly open, there is an open set \(U\) such that \(U \subset A \subset \text{scl } U\). By Lemma 1, \(\text{scl } U = U \cup \text{int } (\text{cl } U) = \text{int } (\text{cl } U)\). Hence \(U \subset A \subset \text{int } (\text{cl } U)\), and consequently \(\text{int } (\text{cl } (\text{int } A)) = \text{int } (\text{cl } U)\). Thus we have that \(A \subset \text{int } (\text{cl } (\text{int } A))\), so that \(A \in \mathcal{F}^*\).

Conversely, if \(A \in \mathcal{F}^*\), we have that \(\text{int } A \subset A \subset \text{int } (\text{cl } (\text{int } A))\). So if \(U = \text{int } A\) we have \(U \subset A \subset \text{int } (\text{cl } U)\), and Lemma 1 implies \(U \subset A \subset \text{scl } U\), so that \(A\) is feebly open.

**Lemma 2**—Let \(x\) be a point of \((X, \mathcal{F})\). Then either \(\{x\}\) is nowhere dense or \(\{x\} \subset \text{int } (\text{cl } \{x\}) = \text{scl } \{x\}\).

**Proof**: Suppose that \(\{x\}\) is not nowhere dense. Then \(\text{int } (\text{cl } \{x\}) \neq \emptyset\), and so \(x \in \text{int } (\text{cl } \{x\})\). Lemma 1 implies that \(\text{scl } \{x\} = \{x\} \cup \text{int } (\text{cl } \{x\}) = \text{int } (\text{cl } \{x\})\). Thus \(\{x\} \subset \text{int } (\text{cl } \{x\}) = \text{scl } \{x\}\).

2. \(R_0\) **Type Properties**

**Definition 1**—A space \((X, \mathcal{F})\) is \(R_0\) if for each \(U \in SO (X, \mathcal{F}), x \in U\) implies \(\text{scl } \{x\} \subset U\).
Maheshwari and Prasad introduced this class of spaces and used the term \((R_0)\) spaces. The class of feebly \(R_0\) spaces was studied by Maheshwari and Tapi.

**Definition 2**—A space \((X, \mathcal{F})\) is feebly \(R_0\) if for each feebly open set \(G, x \in G\) implies \(\text{fcl}\ \{x\} \subseteq G\).

Lemma 1 provides an immediate proof of the following result.

**Proposition 2**—A space \((X, \mathcal{F})\) is semi-\(R_0\) if and only if for each \(U \subseteq SO(X, \mathcal{F})\), \(x \in U\) implies \(\text{int}\ (\text{cl}\ \{x\}) \subseteq U\).

Our next result is proved by appealing to Proposition 1. It shows that if we re-topologize the space appropriately the concept of feebly \(R_0\) reduces to the familiar concept of \(R_0\).

**Proposition 3**—\((X, \mathcal{F})\) is feebly \(R_0\) if and only if \((X, \mathcal{F}^\ast)\) is \(R_0\).

**Proposition 4**—

(i) If \((X, \mathcal{F})\) is feebly \(R_0\), then it is semi-\(R_0\).

(ii) If \((X, \mathcal{F})\) is \(R_0\), then it is feebly \(R_0\).

**Proof:** (i) We first show that \(R_0\) implies semi-\(R_0\). Let \(V \subseteq SO(X, \mathcal{F})\) and \(x \in V\). There is an open set \(U\) such that \(U \subseteq V \subseteq \text{cl}\ U\). Suppose \(x \in U\). Since \((X, \mathcal{F})\) is \(R_0\), \(\text{cl}\ \{x\} \subseteq U\) and hence \(\text{scl}\ \{x\} \subseteq U \subseteq V\). Now suppose that \(x \in V \setminus U \subseteq \text{cl}\ U \setminus U\). Then \(\text{int}\ (\text{cl}\ \{x\}) = \emptyset\) and \(\text{scl}\ \{x\} \subseteq V\). Hence \((X, \mathcal{F})\) is semi-\(R_0\).

Now suppose that \((X, \mathcal{F})\) is feebly \(R_0\). By Proposition 3, \((X, \mathcal{F}^\ast)\) is \(R_0\) and so is semi-\(R_0\) by the argument above. But \(SO(X, \mathcal{F}^\ast) = SO(X, \mathcal{F})\), so that \((X, \mathcal{F})\) is semi-\(R_0\).

(ii) Let \(x \in U \subseteq \mathcal{F}^\ast\). By Lemma 2, \(\{x\}\) is nowhere dense or \(\{x\} \subseteq \text{int}\ (\text{cl}\ \{x\}) = \text{scl}\ \{x\}\). If \(\{x\}\) is nowhere dense, \(\mathcal{F}^\ast \cap \{x\} = \{x\} \subseteq U\). If \(\{x\} \subseteq \text{int}\ (\text{cl}\ \{x\}) = \text{scl}\ \{x\}\), then \(\mathcal{F} \cap \{x\} \subseteq \text{int}\ (\text{cl}\ \{x\})\) since \((X, \mathcal{F})\) is \(R_0\). But \(\mathcal{F}^\ast \cap \{x\} \subseteq \mathcal{F} \cap \{x\}\), so that \(\mathcal{F}^\ast \cap \{x\} \subseteq \text{scl}\ \{x\}\). By part (i) \((X, \mathcal{F})\) is semi-\(R_0\), and \(U \subseteq SO(X, \mathcal{F})\) implies \(\text{scl}\ \{x\} \subseteq U\). Hence \(\mathcal{F}^\ast \cap \{x\} \subseteq U\). Thus \((X, \mathcal{F}^\ast)\) is \(R_0\), so that by Proposition 3 \((X, \mathcal{F})\) is feebly \(R_0\).

We now provide examples to show that the converses of Proposition 4 are false.

**Example 1**—Let \(X = \{a, b, c\}\) and \(\mathcal{F}\) be the topology \(\{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}\) on \(X\). Then \((X, \mathcal{F})\) is semi-\(R_0\), since \(\text{scl}\ \{x\} = \{x\}\) for each point \(x \in X\). Moreover, \(\mathcal{F}^\ast = \mathcal{F}\) and \((X, \mathcal{F}^\ast)\) is not \(R_0\), so that \((X, \mathcal{F})\) is not feebly \(R_0\).

**Example 2**—Let \(X\) be an infinite set and \(p\) be a fixed point of \(X\). We define a topology \(\mathcal{G}\) on \(X\) as follows: for \(G \subseteq X, G \in \mathcal{G}\) if (i) \(G = \emptyset\) or \(G = X\) or (ii) \(G \subseteq X \setminus \{p\}\) and \(X \setminus G\) is finite. Then \((X, \mathcal{F})\) is not \(T_1\), since for any point \(x\) distinct from \(p\), the only open set containing \(p\), namely \(X\), contains \(x\). On the other hand, \((X, \mathcal{F}^\ast)\) is \(T_1\). For let \(a, b \in X, a \neq b\). If \(a, b \in X \setminus \{p\}\), then \(X \setminus \{b, p\}\)
and \( X - \{a, p\} \) are \( \mathcal{S} \) open sets, and hence \( \mathcal{S}^* \) open sets, separating \( a \) and \( b \) appropriately. If one of \( a \) and \( b \) is the point \( p \), say \( b = p \), then \( X - \{p\} \) is a \( \mathcal{S} \) open set and hence \( \mathcal{S}^* \) open. However, \( X - \{a\} \) is not \( \mathcal{S} \) open, but since \( \mathcal{S} \) int (\( \mathcal{S} \) cl (\( \mathcal{S} \) int (\( X - \{a\}\))) = \( \mathcal{S} \) int (\( \mathcal{S} \) cl (\( X - \{a, p\}\))) = \( X \) we have that \( X - \{a\} \) is an \( \alpha \)-set. Hence \( (X, \mathcal{S}^*) \) is \( T_1 \). Now \( (X, \mathcal{S}) \) is not \( R_0 \) (since it is \( T_0 \) but not \( T_1 \)) while \( (X, \mathcal{S}^*) \) is \( R_0 \) since it is \( T_1 \). Thus \( (X, \mathcal{S}) \) is feebly \( R_0 \).

3. \( T_0 \) Type Properties

Maheshwari and Prasad introduced the class of semi-\( T_0 \) spaces, while Maheshwari and Tapi considered feebly \( T_0 \) spaces.

**Definition 3**—A space \( (X, \mathcal{S}) \) is semi-\( T_0 \) (feebly \( T_0 \)) if for each pair of distinct points \( x \) there is a semi-open (feebly open) set in \( (X, \mathcal{S}) \) containing one point but not the other.

Examples have been given in Maheshwari and Prasad and Maheshwari and Tapi to show that these are weaker notions than the \( T_0 \) property in the sense that \( (X, \mathcal{S}) \) can be semi-\( T_0 \) or feebly \( T_0 \) without being \( T_0 \), while \( T_0 \) implies both semi-\( T_0 \) and feebly \( T_0 \). Examples have not been provided to distinguish between semi-\( T_0 \) and feebly \( T_0 \) spaces. Our next result shows that no such spaces exist.

**Proposition 5**—\( (X, \mathcal{S}) \) is semi-\( T_0 \) if and only if \( (X, \mathcal{S}^*) \) is \( T_0 \) if and only if \( (X, \mathcal{S}) \) is feebly \( T_0 \).

**Proof**: By Proposition 1, \( (X, \mathcal{S}) \) is feebly \( T_0 \) if and only if \( (X, \mathcal{S}^*) \) is \( T_0 \). Let \( (X, \mathcal{S}) \) be semi-\( T_0 \), \( x, y \in X \) and \( x \neq y \). Let \( U \in SO \, (X, \mathcal{S}) \) such that \( y \in U \) and \( x \notin U \). Then \( y \notin \mathcal{S} \) cl \( \{x\} \). By Lemma 2, either \( \{x\} \) is nowhere dense or \( \{x\} \subseteq \mathcal{S} \) int (\( \mathcal{S} \) cl \( \{x\} \)) = \( \mathcal{S} \) cl \( \{x\} \). If \( \{x\} \) is nowhere dense then \( X - \{x\} \in \mathcal{S}^* \), so there is an \( \alpha \)-set containing \( y \) but not \( x \). If \( \{x\} \subset \mathcal{S} \) int (\( \mathcal{S} \) cl \( \{x\} \)) = \( \mathcal{S} \) cl \( \{x\} \), then \( y \notin \mathcal{S} \) int (\( \mathcal{S} \) cl \( \{x\} \)). So there is a regular open set containing \( x \) but not \( y \). Thus \( (X, \mathcal{S}^*) \) is \( T_0 \).

Conversely, if \( (X, \mathcal{S}^*) \) is \( T_0 \), it is semi-\( T_0 \). Thus \( (X, \mathcal{S}) \) is semi-\( T_0 \) since \( SO(X, \mathcal{S}) = SO(X, \mathcal{S}^*) \).

4. \( T_D \) Type Properties

Two different generalizations of the \( T_D \) separation property using semi-open sets have appeared. Recall that a topological space \( (X, \mathcal{S}) \) is said to be a \( T_D \) space if the derived set \( \mathcal{S} \) cl \( \{x\} - \{x\} \) is closed, for each point \( x \in X \). Arya and Bhamini have defined \( (X, \mathcal{S}) \) to be semi-\( T_D \) if the derived set \( \mathcal{S} \) cl \( \{x\} - \{x\} \) is semi-closed, for each point \( x \in X \). Dube and Sengar say that \( (X, \mathcal{S}) \) is semi-\( T_D \) if the semi-derived set \( \mathcal{S} \) cl \( \{x\} - \{x\} \) is semi-closed, for each point \( x \in X \). Our next result shows that these concepts are equivalent to each other, and to \( (X, \mathcal{S}^*) \) is \( T_D \).
Proposition 6—The following are equivalent for a topological space \((X, \mathcal{G})\).

(a) \(\text{scl } \{x\} - \{x\}\) is semi-closed for each \(x \in X\).
(b) \(\text{cl } \{x\} - \{x\}\) is semi-closed for each \(x \in X\).
(c) \(\{x\}\) is either open or nowhere dense for each \(x \in X\).
(d) \((X, \mathcal{G}^*)\) is \(T_D\).

Proof: (a) implies (b): Let \(x \in X\). By Lemma 2, either \(\{x\}\) is nowhere dense or \(\{x\} \subseteq \text{int (cl } \{x\}) = \text{scl } \{x\}\). If \(\{x\}\) is nowhere dense, then \(\text{cl } U = X\) where \(U = X - \text{cl } \{x\}\). Since \(U \in \mathcal{G}\) and \(x \in \text{cl } U\), \(U \cup \{x\} \in \text{SO}(X, \mathcal{G})\). Hence \(X - (U \cup \{x\}) = \text{cl } \{x\} - \{x\}\) is semi-closed. On the other hand, if \(\{x\} \subseteq \text{int (cl } \{x\}) = \text{scl } \{x\}\), then \(\text{cl } (\{x\} - \{x\}) = (\text{cl } \{x\} - \{x\})(\text{int (cl } \{x\}) \cup (\text{int (cl } \{x\})) - \{x\}).\) Now \(\text{cl } \{x\} - \text{int (cl } \{x\})\) is closed while \(\text{int (cl } \{x\} - \{x\}) = \text{scl } \{x\} - \{x\}\) which is semi-closed by hypothesis. Thus \(\text{cl } \{x\} - \{x\}\) is semi-closed.

(b) implies (a): Let \(x \in X\). Since \(\text{scl } \{x\} - \{x\} = (\text{cl } \{x\} - \{x\})(\text{int scl } \{x\}) \cap \text{scl } \{x\}\), and \(\text{cl } \{x\} - \{x\}\) is semi-closed by hypothesis, we have \(\text{scl } \{x\} - \{x\}\) is semi-closed.

(a) implies (c): Suppose \(\text{scl } \{x\} = \{x\}\). Then either \(\{x\}\) is nowhere dense, or \(\{x\} = \text{int (cl } \{x\})\) so \(\{x\}\) is open. On the other hand, suppose that \(\{x\}\) is a proper subset of \(\text{scl } \{x\}\). Using \(\text{int (cl } \{x\}) = \text{scl } \{x\}\) we obtain \((\text{int cl } \{x\}) \cap (X - \text{cl int } \{x\}) = (\text{int cl } \{x\}) \cap \text{cl } (X - \{x\}) = \text{int cl } (\text{scl } \{x\} = \text{scl } \{x\} - \{x\} = \text{int cl } \{x\} \cap (X - \{x\}))\) which is closed. Thus \(\text{int cl } \{x\}\), and \(\{x\}\) is open.

(c) implies (d): Let \(x \in X\). By (c), \(\{x\}\) is either nowhere dense or open. If \(\{x\}\) is nowhere dense then \(\mathcal{G}^*\text{ cl } \{x\} = \{x\}\), so the \(\mathcal{G}^*\) derived set of \(\{x\}\) is empty and so \(\mathcal{G}^*\) closed. If \(\{x\}\) is open, then \(\mathcal{G}^*\text{ cl } \{x\} - \{x\} = \mathcal{G}^*\text{ cl } \{x\} \cap (X - \{x\})\) is \(\mathcal{G}^*\) closed. Thus \((X, \mathcal{G}^*)\) is \(T_D\).

(d) implies (a), Let \((X, \mathcal{G}^*)\) be a \(T_D\) space and \(x \in X\). Then \(\mathcal{G}^*\text{ cl } \{x\} - \{x\}\) is \(\mathcal{G}^*\) closed, and therefore semi-closed. Since (b) implies (a), \(\mathcal{G}^*\text{ scl } \{x\} - \{x\}\) is semi-closed in \((X, \mathcal{G}^*)\). But \(\text{SO} (X, \mathcal{G}) = \text{SO} (X, \mathcal{G}^*)\) implies that \(\mathcal{G}\text{ scl } \{x\} = \{x\}\) is semi-closed, as desired.

5. \(T_1\) Type Properties

The class of semi-\(T_1\) spaces was defined by Maheshwari and Prasad\(^7\), while the feebly \(T_1\) spaces were considered by Maheshwari and Tapi\(^8\).

Definition 4—A space \((X, \mathcal{G})\) is semi-\(T_1\) (feebly \(T_1\)) if for each pair of distinct points \(x, y\) in \(X\) there is a semi-open (feebly open) set containing \(x\) but not \(y\).
Proposition 7—\((X, \mathcal{F})\) is semi-\(T_1\), if and only if each singleton is nowhere dense or regular open.

**Proof**: Let \(x \in X\). By [7, Theorem 4.1] \(\{x\}\) is semi-closed, so \(\text{scl} \{x\} = \{x\}\). Thus Lemma 2 implies that \(\{x\}\) is nowhere dense or \(\{x\} \in \text{RO} (X, \mathcal{F})\).

The converse is clear, since nowhere dense sets and regular open sets are semi-closed.

Proposition 8—\((X, \mathcal{F})\) is feebly \(T_1\), if and only if \((X, \mathcal{F}^*)\) is \(T_1\), if and only if each singleton is nowhere dense or clopen.

**Proof**: By Proposition 1, \((X, \mathcal{F})\) is feebly \(T_1\) if and only if \((X, \mathcal{F}^*)\) is \(T_1\).

Let \((X, \mathcal{F}^*)\) be \(T_1\) and \(x \in X\). Then \(\mathcal{F}^* \text{cl} \{x\} = \{x\}\), so that \(\text{cl} (\text{int} (\text{cl} \{x\})) \subseteq \{x\}\). By Lemma 2, \(\{x\}\) is nowhere dense or \(\{x\} \subset \text{int} (\text{cl} \{x\})\). If \(\{x\} \subseteq \text{int} (\text{cl} \{x\})\), then \(\{x\} = \text{cl} (\text{int} (\text{cl} \{x\}))\) and \(\{x\} = \text{int} (\text{cl} \{x\})\), so that \(\{x\}\) is clopen. The converse is clear, since nowhere dense sets and clopen sets are \(\mathcal{F}^*\) closed.

As an immediate corollary to the last two results we have

Proposition 9—It \((X, \mathcal{F})\) is feebly \(T_1\), then it is semi-\(T_1\).

To see that the converse of Proposition 9 is false consider the space \((X, \mathcal{F})\) of Example 1.

6. \(T_2\) Type Properties

Noiri [12, Proof of Corollary 4.7] has given an elegant proof of the fact that \((X, \mathcal{F})\) is \(T_2\) if and only if \((X, \mathcal{F}^*)\) is \(T_2\). Thus the notion of feebly \(T_2\) coincides with the usual Hausdorff property. Maheshwari and Prasad have defined \((X, \mathcal{F})\) to be semi-\(T_2\) if for each pair of distinct points \(x, y \in X\) there are disjoint semi-open sets \(A, B \subseteq X\) such that \(x \in A\), \(y \in B\). Note that the space \((X, \mathcal{F})\) of Example 1 is semi-\(T_1\) and hence \((X, \mathcal{F}^*)\) is not \(T_2\).

Cameron introduced the concept of regular semi-open sets and used it to characterize \(S\)-closed topological spaces. A subset \(A\) of a space \(X\) is said to be regular semi-open if there is regular open set \(U\) of \(X\) such that \(U \subseteq A \subseteq \text{cl} U\). We denote the collection of regular semi-open sets in \((X, \mathcal{F})\) by \(\text{RSO} (X, \mathcal{F})\).

Lemma 3—A subset \(A\) of a space \((X, \mathcal{F})\) is regular semi-open if and only if \(A\) is semi-open and semi-closed.

**Proof**: Let \(A\) be a regular semi-open subset of \(X\). Then there is a regular open subset \(U\) of \(X\) such that \(U \subseteq A \subseteq \text{cl} U\). This implies that \(U = \text{int} (\text{cl} U) = \text{int} (\text{cl} A)\). Therefore \(\text{int} (\text{cl} A) \subseteq A\) and \(A\) is semi-closed. Clearly, \(A\) is also semi-open.
Conversely, let $A$ be a semi-open and semi-closed subset of $X$. Then $\text{int} (\text{cl} A) \subseteq A \subseteq \text{cl} (\text{int} A)$ and consequently $\text{int} (\text{cl} A) \subseteq A \subseteq \text{cl} (\text{int} (\text{cl} A))$. Since $\text{int} (\text{cl} A)$ is regular open, $A$ is regular semi-open.

It is not difficult to observe that the semi-closure of a semi-open set is semi-closed and semi-open, and so regular semi-open by Lemma 3. Now the following characterization of semi-$T_2$ spaces is readily established.

**Proposition** 10.—A space $X$ is semi-$T_2$ if and only if for each pair of distinct points $x, y$ in $X$ there is a regular semi-open set $U$ containing $x$ but not $y$ or containing $y$ but not $x$.

Maheshwari *et al.* defined regular semi-$T_i$ spaces ($i = 0, 1, 2$) were replacing the expression “open set” in the definition of $T_i$ spaces ($i = 0, 1, 2$) with “regular semi-open set” and it was shown that regular semi-$T_2$ implies regular semi-$T_1$ which implies regular semi-$T_0$ and that regular semi-$T_2$ implies semi-$T_2$.

However, no examples were provided to show that these are different classes of spaces. By Proposition 10 it follows that these implications are reversible, and that these notions define the same class of spaces.

A topological property $P$ is said to be semi-regular provided that $(X, \mathcal{F})$ has property $P$ if and only if $(X, \mathcal{F})$ has property $P$.

**Lemma** 4.—$\text{RSO} (X, \mathcal{F}) = \text{RSO} (X, \mathcal{F})$.

**Proof**: Since $\text{SO} (X, \mathcal{F}) \subseteq \text{SO} (X, \mathcal{F})$ and $\text{SC} (X, \mathcal{F}) \subseteq \text{SC} (X, \mathcal{F})$ we have from Lemma 3 that $\text{RSO} (X, \mathcal{F}) \subseteq \text{RSO} (X, \mathcal{F})$.

Conversely, let $A \in \text{RSO} (X, \mathcal{F})$. Then there is a set $U \in \text{RO} (X, \mathcal{F})$ such that $U \subseteq A \subseteq \text{cl} U$. Now $\text{RO} (X, \mathcal{F}) = \text{RO} (X, \mathcal{F})$ and $\mathcal{F} \text{cl} U = \mathcal{F} \text{cl} U$, so that $U \in \text{RSO} (X, \mathcal{F})$.

Our next result follows easily from Proposition 10 and Lemma 4.

**Proposition** 11.—Semi-$T_2$ is a semi-regular property.

We define a bijection $f : (X, \mathcal{F}) \rightarrow (Y, S)$ to be a regular semihomeomorphism if $f (U) \in \text{RSO} (Y, S)$ for each $U \in \text{RSO} (X, \mathcal{F})$ and $f^{-1} (V) \in \text{RSO} (X, \mathcal{F})$ for each $V \in \text{RSO} (Y, S)$. The following result will enable us to show that semiregular properties are shared by $(X, \mathcal{F})$ and $(X, \mathcal{F}^*)$. Recall that a subset of a space is regular closed if its complement is regular open.

**Proposition** 12.—A bijection $f : (X, \mathcal{F}) \rightarrow (Y, S)$ is a regular-semihomeomorphism if and only if $f : (X, \mathcal{F}) \rightarrow (Y, S)$ is homeomorphism.

**Proof**: To establish that the condition is necessary, let $V$ be a regular closed subset of $(Y, S)$. Then $V \in \text{RSO} (Y, S)$ and $f^{-1} (V) \in \text{RSO} (X, \mathcal{F})$. Suppose that there exists an $x \in \text{cl} f^{-1} (V) - f^{-1} (V)$. Since $f^{-1} (V) \cup \{ x \} \in \text{RSO} (X, \mathcal{F})$, $f (f^{-1} (V) \cup \{ x \}) = V \cup \{ f (x) \} \in \text{RSO} (Y, S)$. Therefore
$V \cup \{f(x)\} \subset \text{cl } (\text{int } (V \cup \{f(x)\})) \subset \text{cl } (V \cup \text{int } \{f(x)\}) \subset V \cup \text{cl } (\text{int } \{f(x)\})$. Since $f(x) \in V$, $\{f(x)\} \subset \text{cl } (\text{int } \{f(x)\})$ and hence $\{f(x)\} \in S$. On the other hand

$$V \cup \text{scl } \{f(x)\} = \text{scl } V \cup \text{scl } \{f(x)\} \subset \text{scl } (V \cup \{f(x)\}) = V \cup \{f(x)\}.$$ 

Since $f(x) \in V$ and $V \in SO(Y, S)$, $\text{scl } \{f(x)\} \cap V = \phi$. Therefore $\text{scl } \{f(x)\} \subset \{f(x)\}$. Since $\{f(x)\}$ is open and semi-closed, $\{f(x)\} \in RO(Y, S)$. This implies that $\{x\} = f^{-1}(\{f(x)\}) \in RSO(X, \mathcal{F})$ and consequently $x \in RO(X, \mathcal{F})$. This contradicts the assumption that $x \in \text{cl } f^{-1}(V) - f^{-1}(V)$. Hence $f^{-1}(V)$ is closed and since $f^{-1}(V) \in SO(Y, S)$, $f^{-1}(V)$ is regular closed. Now it is easily established that $f : (X, \mathcal{F}) \to (Y, S_a)$ is a homeomorphism.

Conversely, since homeomorphisms are regular-semihomeomorphisms, and compositions of regular-semihomeomorphisms are regular-semihomeomorphisms, the sufficiency follows by Lemma 4.

**Proposition 13**—Semiregular properties are shared by $(X, \mathcal{F})$ and $(X, \mathcal{F}^a)$.

**Proof**: Proposition 12 it follows that semiregular properties are precisely the properties preserved under regular-semihomomorphisms. On the other hand, $(X, \mathcal{F})$ and $(X, \mathcal{F}^a)$ are regularly semi homeomorphic since $RSO(X, \mathcal{F}) = RSO(X, \mathcal{F}^a)$.

By Proposition 13 we obtain a wide class of topological properties shared by $(X, \mathcal{F})$ and $(X, \mathcal{F}^a)$. For example, the mentioned result that $(X, \mathcal{F})$ is $T_1$ if and only if $(X, \mathcal{F}^a)$ is $T_2$ follows from the well known fact that $T_2$ is a semi-regular property.

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**References**