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COMMON FIXED POINT OF MAPPINGS IN A UNIFORMLY CONVEX BANACH SPACE WITH A NEW FUNCTIONAL INEQUALITY

DEVI PRASAD

Mathematics Group, Birla Institute of Technology and Science, Pilani 333031, *Rajasthan*

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In this paper a common fixed point theorem of two maps f and g on a non-empty closed subset k of a uniformly convex Banach space is proved. The maps f and g satisfy the inequality

$$\|fx - gy\|^2 \leqslant \phi(\|x - fx\|\ \|y - gy\|,\ \|x - gy\|\ \|y - fx\|,$$
$$\|x - fx\|\ \|x - gy\|,\ \|y - fx\|\ \|y - gy\|)$$
for all $x, y \in k$

where ϕ is an upper semi-continuous function from R_+^4 to R_+ satisfying

(i) $\phi(t, 0, \alpha t, 0) \leqslant \beta t$ and $\phi(t, 0, 0, \alpha t) \leqslant \beta t$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$;

(ii) $\phi(t, t, t, t) < t$.

A new common fixed point theorem for a pair of self-mappings of closed convex subset in a uniformly convex Banach space has been proved by Bose (1978). This generalizes the results of several authors e. g. Singh and Meade (1977), Srivastav and Gupta (1971), Wong (1973), Iseki (1974), Hussain and Sehgal (1975). In this paper a common fixed point theorem similar to that of Bose with a different functional inequality for self-mappings of a closed subset of a uniformly convex Banach space is proved.

Let R_+ be the non-negative reals. Let ϕ be a mappings from R_+^4 to R_+ which is upper semi-continuous and non-decreasing in each variable. The theorem proved is stated below :

Theorem—Let X be a uniformly convex Banach space and K be a non-empty closed subset of X. Let f and g be two mappings from K into K such that for all x, y in K,

$$\|fx - gy\|^2 \leqslant \phi\,(\|x - fx\|\ \|y - gy\|,\ \|x - gy\|\ \|y - fx\|,$$
$$\|x - fx\|\ \|x - gy\|,\ \|y - fx\|\ \|y - gy\|) \qquad \ldots(*)$$

and for all $t > 0$,

(i) $\phi\,(t, 0, \alpha t, 0) \leqslant \beta t$ and $\phi\,(t, 0, 0, \alpha t) \leqslant \beta t$

where $\beta = 1$ for $\alpha = 2$ and $\beta < 1$ for $\alpha < 2$;

(ii) $\phi\,(t, t, t, t) < t$.

Then there exists a point $u \in K$ such that

(a) $fu = gu = u$,

(b) u is unique fixed point of each f and g,

(c) for any $x_0 \in K$, the sequence $\{x_n\}$ generated by $x_1 = fx_0$, $x_2 = gx_1$, $x_3 = fx_2, \ldots\ldots$ converges strongly to u.

For the proof of this theorem, we need the following. The modulus of convexity of a Banach space X is a function $\delta : (0, 2] \to (0, 1]$ defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\|,\ x, y \in X,\ \|x\| = \|y\| = 1,\ \|x - y\| \geqslant \epsilon\right\}.$$

It is known (Iseki 1974) that if X is uniformly convex then δ is strictly increasing,

$$\lim_{\epsilon \to 0} \delta(\epsilon) = 0 \text{ and } \delta(2) = 1.$$

Denoting the inverse of δ by η, we therefore note that $\eta(\epsilon) < 2$ for $t < 1$. The following lemma has been proved by Goebel *et al.* (1973).

Lemma 1—Let X be a uniformly convex Banach space. Let B_R denote the closed ball in X centered at the origin with radius $R > 0$, if $x_1, x_2, x_3 \in B_R$,

$$\|x_1 - x_2\| \geqslant \|x_2 - x_3\| \geqslant d > 0$$

and

$$\|x_2\| \geqslant \left(1 - \frac{1}{2}\,\delta\left(\frac{d}{R}\right)\right)R$$

then

$$\|x_1 - x_3\| \leqslant \eta\left(1 - \frac{1}{2}\,\delta\left(\frac{d}{R}\right)\right)\|x_1 - x_2\|.$$

PROOF OF THEOREM : Let $x_0 \in K$ and x_n be the sequence as defined in (c), let d_n stand for $\|x_n - x_{n+1}\|$. We first show

$$d_n \leqslant d_{n-1},\ n = 1, 2, 3,\ldots \qquad \ldots(1)$$

It follows from (*) that for any positive integer n,

$$d_n^2 \leqslant \begin{cases} \phi\,(d_{n-1}\,d_n,\ 0,\ d_{n-1}\,(d_{n-1} + d_n),\ 0), \\ \qquad\qquad \text{if } n \text{ is odd, and} \qquad \ldots(2) \\ \phi\,(d_n\,d_{n-1},\ 0,\ 0,\ (d_{n-1} + d_n)\,d_n), \\ \qquad\qquad \text{if } n \text{ is even.} \qquad \ldots(3) \end{cases}$$

Suppose for some n, $d_n > d_{n-1}$, then $d_{n-1} + d_n = \alpha\, d_n$ with some $\alpha < 2$ and ϕ being non-decreasing in each variable,

$$d_n^2 \leqslant \begin{cases} \phi\,(d_n^2,\ 0,\ \alpha d_n^2,\ 0) \text{ for } n \text{ odd.} \\ \phi\,(d_n^2,\ 0,\ 0,\ \alpha d_n^2) \text{ for } n \text{ even.} \end{cases}$$

In either case (i) gives

$$d_n^2 \leqslant \beta\, d_n^2 < d_n^2 \quad \because\ \beta < 1$$

which is a contradiction to for some n, $d_n > d_{n-1}$. This proves (1) and therefore $d_n \to d$ (say) $\geqslant 0$.

Now we show $d = 0$.

If not suppose $d > 0$. Assume without loss of generality that $0 \in K$.

Let $R' = \lim\limits_{n\to\infty} \sup \|x_n\|$.

We can assume $R' \neq 0$. Let $R > R'$ be chosen such that

$$R\left(1 - \frac{1}{2}\ \delta\left(\frac{d}{R}\right)\right) < R'.$$

Clearly there is a sequence $\{n_i\}$, $i = 0, 1,$ ..... of positive integers such that for any $j \in \{n_i\}$,

$$\|x_j\| \geqslant R\left(1 - \frac{1}{2}\ \delta\left(\frac{d}{R}\right)\right)$$

while for all $n \geqslant n_0$, $\|x_n\| \leqslant R$,

It therefore follows from Lemma 1, that for any $j \in \{n_i\}$,

$$\|x_{j-1} - x_{j+1}\| \leqslant \eta\left(1 - \frac{1}{2}\delta\left(\frac{d}{R}\right)\right)\|x_{j-1} - x_j\|$$

$$= \alpha_1 \|x_{j-1} - x_j\| \qquad \ldots(4)$$

where $\alpha_1 = \eta\,(1 - 1/2\,\delta\,(d/R)) < 2$ because of uniform convexity. Since $d_j \leqslant d_{j-1}$ it follows from (2), (3) and (4) that

$$d_j^2 \leqslant \begin{cases} \phi\,(d_{j-1}^2,\ 0,\ \alpha_1 d_{j-1}^2,\ 0) \text{ for odd } j \in \{n_i\} \\ \phi\,(d_{j-1}^2,\ 0,\ 0,\ \alpha_1 d_{j-1}^2). \text{ for even } j \in \{n_i\}. \end{cases}$$

In either case by using (i), we get

$$d_j^2 \leqslant \beta\, d_{j-1}^2 \text{ for some } \beta < 1.$$

This implies that $d < \sqrt{\beta}\, d < d$, a contradiction. Hence $d = 0$.

Since $d\,(x_{n-1}, x_n) \downarrow 0$, it follows by an identical proof in Hussain and Sehgal (1975) that x_n is a Cauchy sequence.

X is Banach space, therefore $\{x_n\}$ converges to a point u in X and $\{x_n\}$ is in K and K is closed, so $u \in K$.

Now we show that $f(u) = g\,(u) = u$.

Let $\|fu - u\| = r > 0$, there exists N such that for all $n \geqslant N$ (some chosen integer)

$$\|x_{2n} - u\| \leqslant \frac{r}{8} \text{ and } \|x_{2n} - x_{2n-1}\| \leqslant \frac{r}{8} \text{ since } \{x_n\} \to u.$$

Now

$$\|u - x_{2n-1}\| \leqslant \|u - x_{2n}\| + \|x_{2n} - x_{2n-1}\| < \frac{r}{8} + \frac{r}{8} = \frac{r}{4}.$$

and

$$\|x_{2n-1} - fu\| \leqslant \|x_{2n-1} - u\| + \|u - fu\| < \frac{r}{4} + r = \frac{5r}{4}.$$

$$r = \|fu - u\| < \frac{r}{8} + \|fu - x_{2n}\|. \qquad \ldots(5)$$

Now

$$\|fu - x_{2n}\|^2 = \|fu - gx_{2n-1}\|^2$$

$$\leqslant \phi\,(\|u - fu\|\,\|x_{2n-1} - gx_{2n-1}\|,\ \|u - gx_{2n-1}\|\,\|x_{2n-1} - fu\|,$$

$$\|u - fu\|\,\|u - gx_{2n-1}\|,\ \|x_{2n-1} - fu\|\,\|x_{2n-1} - gx_{2n-1}\|)$$

Note : In x_{j-1} and x_{j+1}; $j-1$ and $j+1$ are subscripts.

$$\leqslant \phi\,(\|u - gu\|\,\|x_{2n-1} - x_{2n}\|,\ \|u - x_{2n}\|\,\|x_{2n-1} - fu\|,$$

$$\|u - fu\|\,\|u - x_{2n}\|,\ \|x_{2n-1} - fu\|\,\|x_{2n-1} - x_{2n}\|)$$

$$\leqslant \phi\left[r.\ \frac{r}{8}, \frac{r}{8}.\frac{5r}{8}, r.\ \frac{r}{8}, \frac{5r}{4}.\frac{r}{8}\right]$$

$$\leqslant \phi\left(\frac{r^2}{8}, \frac{5r^2}{32}, \frac{r^2}{8}, \frac{5r^2}{32}\right)$$

$$\leqslant \phi\left(\frac{5r^2}{32}, \frac{5r^2}{32}, \frac{5r^2}{32}, \frac{5r^2}{32}\right)$$

$$< \frac{5r^2}{32} \text{ from (ii).}$$

$$\text{i.e. } \|fu - x_{2n}\| < \frac{\sqrt{10}}{8}\, r.$$

Now from (5) and above

$$r < \frac{r}{8} + \frac{\sqrt{10}}{8}\, r$$

$$< \left(\frac{1}{8} + \frac{\sqrt{10}}{8}\right) r = \frac{1 + \sqrt{10}}{8}\, r$$

i.e. $r < \dfrac{1 + \sqrt{10}}{8}\, r$, a contradiction

hence $r = 0$ i.e. $\|fu - u\| = 0$

$$\therefore fu = u.$$

Similarly, we get $gu = u$.

Hence $fu = gu = u$.

Now to prove uniqueness of the fixed points of f and g Let v be another common fixed point of f and g i.e. $fu = u$ and $gv = v$.

Let

$$\|u - v\| \neq 0.$$

$$\|u - v\|^2 = \|fu - gu\|^2$$

$$\leqslant \phi\,(\|u - fu\|\,\|v - gv\|,\ \|u - gv\|\,\|v - fu\|,$$

$$\|u - fu\|\,\|u - gv\|,\ \|v - gu\|\,\|v - gv\|)$$

$$\leqslant \phi\,(0, \|u - v\|^2, 0, 0)$$

$$\leqslant \phi\,(\|u - v\|^2, \|u - v\|^2, \|u - v\|^2, \|u - v\|^2)$$

$$< \|u - v\|^2$$

$$\therefore \|u - v\|^2 < \|u - v\|^2, \text{ a contradiction.}$$

so $\|u - v\| = 0$ i.e. $u = v$.

This proves uniqueness.

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