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A FIXED POINT THEOREM FOR COMMUTING MAPS

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A fixed point theorem has been established for a pair of commutative mappings of a metric space. This result leads to a direct generalization of the fixed point theorems of Rhoades (1977) and Das and Naik (1979).

Let (X, d) be a metric space. If X is complete and g, a self-map of X, satisfies

$$d(gx, gy) \leqslant \alpha . \max \{d(x, y), d(x, gx), d(y, gy), d(x, gy), d(y, gx)\} \quad \ldots(1)$$

for every $x, y \in X$ and for some $\alpha \in [0, 1)$, then it is known that g has a unique fixed point in X (Ćirić 1974). Das and Naik (1979) have generalized this result following the spirit of Jungck (1976) and hence established the following fixed point theorem for two commutative mappings.

Theorem (Das and Naik 1979) — Let (X, d) be a complete metric space. Let f be a continuous self-map of X and g be any self-map of X which commutes with f. Further, if $g(X) \subset f(X)$ and

$$d(gx, gy) \leqslant \alpha . \max \{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\} \quad \ldots(2)$$

for every $x, y \in X$ and for some $\alpha \in [0, 1)$, then f and g have a unique common fixed point in X.

If we set $f = I_X$, the identity map on X, in the above theorem, then we observe that the result of Das and Naik (1979) is identical with that of Ćirić (1974). Further, we observe that (2) implies

$$d(gx, gy) < \max \{d(fx, fy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\} \quad \ldots(3)$$

since $\alpha < 1$. This shows that the condition (3) is weaker than (2). In this note it is our aim to establish a sufficient condition which will ensure the existence of a common fixed point of f and g satisfying (3). In this connection we prove the following result.

Theorem — Let (X, d) be a compact metric space. Let f be a self-map of X and g be a continuous self-map of X which commutes with f. Further, if $g(X) \subset f(X)$ and f, g satisfy (3) for every x, y $(x \neq y) \in X$, then f and g have a unique common fixed point in X.

PROOF : Since g is continuous and X is compact, then $g(X)$ is compact. Indeed, each iterate $g^n(X)$ is compact. Further, $g^{n+1}(X) \subset g^n(X)$ for all n. Then $\{g^n(X)\}_{n=1}^{\infty}$ is a family of compact sets with a finite intersection property and so will have a non-void intersection $A = \bigcap_{n=1}^{\infty} g^n(X)$. Hence A is a compact subset of X and is mapped onto itself by g.

We now show that A is mapped into itself by f. Let $x_0 \in A$. Then $x_0 \in g^n(X)$ for all n. This implies $fx_0 \in f g^n(X) = g^n f(X)$ for all n, because g commutes with f. Further, since $f(X) \subset X$, then $fx_0 \in g^n(X)$ for all n, implying $f(A) \subseteq A$.

Next, we show that A is a singleton. If it is not so, we then define a function $h : A \times A \to R^+$ (the non-negative reals) by $h(x, y) = d(x, y)$. Since $A \times A$ is compact and h is continuous, there exist points x', y' at which h attains its maximum. Hence δ, the diameter of A, is $h(x', y') = d(x', y')$. Because g maps A onto itself, there are points x'', y'' with $x' = gx''$, $y' = gy''$. Now from (3) we get

$$\begin{aligned}\delta &= d(x', y') = d(gx'', gy'') \\ &< \max\{d(fx'', fy''), d(fx'', gx''), d(fy'', gy''), d(fx'', gy''), d(fy'', gx'')\} \\ &\leqslant \delta\end{aligned}$$

which is a contradiction. Thus A is a singleton $\{a\}$, say. Hence a is a common fixed point of f and g. The uniqueness of the fixed point follows readily from (3). This completes the proof.

It may be noted that the above theorem reduces to that of Rhoades (1977) if we set $f = I_X$, provided the right-hand side of inequality is non-zero.

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