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ON OBTAINING MULTILATERAL GENERATING FUNCTIONS

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By exploiting the properties of the operator $\theta = x^{k+1}\, d/dx$, quite a general class of multilateral generating functions involving the polynomials $\{G_n^{(\alpha)}(x, s, p, k)\}$ of Srivastava and Singhal (1971) and the sequences $\{M_{n_j}(x)\}$ $(j = 1, 2, \ldots, r)$, generated by

$$\sum_0^\infty M_{n_j}(x)\, t^{n_j} = \exp[H_j(x)\, t]\ {}_{B^{(j)}}F_{C^{(j)}}\left[\begin{matrix}(b^{(j)});\\(c^{(j)});\end{matrix} \quad f_j(x)\, t^{q_j}\right]$$

are obtained. Some interesting particular cases are listed.

1. INTRODUCTION

Srivastava and Singhal (1971) defined the following class of polynomials by means of a generalized Rodrigues' formula

$$G_n^{(\alpha)}(x, s, p, k) = \frac{x^{-\alpha-kn}\exp(px^s)}{n!}\,\theta^n[x^\alpha \exp(-px^s)] \qquad \ldots(1.1)$$

where the differential operator θ is given by

$$\theta = x^{k+1}\, d/dx, \ (k \text{ may be any integer}). \qquad \ldots(1.2)$$

In this paper we shall exploit the properties of the operator θ to obtain multilateral generating functions for several classes of higher transcendental functions of mathematical physics. In our analysis, we shall need the following known properties of the operator θ, as given by Srivastava and Singhal (1971, p. 76, eqns. (1.10), eqn. (1.12))

$$\theta^n(x^\alpha) = k^n(\alpha/k)_n\, x^{\alpha+nk} \qquad \ldots(1.3)$$

$$e^{t\theta} f(x) = f(x[1 - x^k t k]^{-1/k}) \qquad \ldots(1.4)$$

Let further the sequences of functions $\{M_{n_j}(x)\}$ $(j = 1, 2, \ldots, r)$ be generated by [see also Section 4 of this paper in which we recall such a generating function introduced by Srivastava (1969, p. 203, eqn. (8)]

$$\sum_{0}^{\infty} M_{n_j}(x)\, t^{n_j} = \exp\left[H_j(x)\, t\right] {}_{B^{(j)}}F_{C^{(j)}}\left[\begin{matrix}(b^{(j)});\\ (c^{(j)});\end{matrix} \quad f_j(x)\, t^{q_j}\right] \qquad \ldots(1.5)$$

where q_j's are any arbitrary integers $H_j(x)$ and $f_j(x)$ are functions of x, $(b^{(j)})$ denotes the sequence of $B^{(j)}$ parameters $b_1^{(j)}, b_2^{(j)}, \ldots b_{B^{(j)}}^{(j)}$ with similar interpretation for $(c^{(j)})$, $j = 1, 2, \ldots, r$.

2. Principal Result

In this section, we prove the following main result.

Main Theorem

For $\{G_n^{(\alpha)}(x, s, p, k)\}$ and $\{M_{n_j}(x)\}$ $(j = 1, 2, \ldots, r)$ defined by (1.4) and (1.5), respectively, we have the multilateral generating relation

$$\sum_{n_1,\ldots,n_r=0}^{\infty} k^{-(m+\Sigma n_i)}\,(m+\Sigma n_i)!\; G_{m+\Sigma n_i}^{(\alpha)}(x, s, p, k) \prod_{j=1}^{r} M_{n_j}(y_j)\, u_j^{n_j}$$

$$= \exp(px^s)\,\{\Delta_r\}^{-m-(\alpha/k)} \sum_{l=0}^{\infty} \frac{(-px^s\,\Delta_r^{-s/k})^l}{l!} \left(\frac{\alpha+sl}{k}\right)_m$$

$$F_{0:\, C^{(1)};\ldots;\, C^{(r)}}^{1:\, B^{(1)};\ldots;\, B^{(r)}}\left(\begin{matrix}\left[\dfrac{\alpha+sl}{k}+m : q_1,\ldots,q_r\right] : [(b^{(1)}) : 1];\ldots;[(b^{(r)}) : 1];\\ \qquad\qquad\qquad\qquad\qquad\qquad : [(c^{(1)}) : 1];\ldots;[(c^{(r)}) : 1];\\ [f_1(y_1)\,(u_1/\Delta_r)^{q_1}],\ldots,[f_r(y_r)\,(u_r/\Delta_r)^{q_r}]\end{matrix}\right) \qquad \ldots(2.1)$$

where

$$\Delta_r = 1 - \sum_{j=1}^{r} u_j H_j(y_j), \quad r = 1, 2, \ldots, \qquad \ldots(2.2)$$

and on the right-hand side we use the notation for the generalized Lauricella function of several variables defined and studied by Srivastava and Daoust (1969, p. 454); see

also Exton (1976, p. 106). This very notation has also been used elsewhere in this article.

As $G_n^{(\alpha)}(x, s, p, k)$ subsume Laguerre and Hermite polynomials and several of their generalizations (see Srivastava and Singhal 1971), and since the sequences of functions $\{M_{n_j}(y_j)\}$ $(j = 1, 2, \ldots, r)$ are of general nature, the multilateral generating function (2.1) yields as particular cases results obtained by other workers; see Patil and Thakare (1978), Srivastava and Singhal (1972a), Thakare (1980), Carlitz (1960), and others.

It may be mentioned that our treatment is formal and, more importantly, that Srivastava (1981, 1982) has proved several general classes of multilinear (and multilateral) generating functions by employing a markedly different approach than ours.

Proof of the Main Theorem

Let $I_r(u_1, \ldots, u_r)$ denote the left-hand side of (2.1). Replace u_j by $u_j k\, x^k$ $(j = 1, 2, \ldots, r)$ and making use of the relations (1.4) and (1.5) we observe that

$$\exp(-px^s)\, x^{\alpha+km}\, k^m\, I_r(u_1 k x^k, \ldots, u_r k\, x^k)$$

$$= \theta^m \prod_{j=1}^{r} \left\{ \sum_{n_j=0}^{\infty} M_{n_j}(y_j)\, (u_j\theta)^{n_j} \right\} x^\alpha \exp(-px^s)$$

$$= \theta^m \exp\left\{ \left(\sum_{j=1}^{r} u_j H_j(y_j) \right) \theta \right\}$$

$$\prod_{j=1}^{r} {}_{B^{(j)}}F_{C^{(j)}}\left[\begin{matrix} (b^{(j)}); \\ (c^{(j)}); \end{matrix} \quad f_j(y_j)\, (u_j\theta)^{q_j} \right] x^\alpha \exp(-px^s)$$

$$= \exp\left\{ \left(\sum_{j=1}^{r} u_j H_j(y_j) \right) \theta \right\}$$

$$\sum_{l, n_1, \ldots, n_r = 0}^{\infty} \frac{(-p)^l}{l!} \left\{ \prod_{j=1}^{r} \frac{[(b^{(j)})]_{n_j} [f_j(y_j)\, u_j^{q_j}]^{n_j}}{[(c^{(j)})]_{n_j}\, n_j!} \right\} \theta^{m+q_1 n_1 + \cdots + q_r n_r}\, x^{\alpha+sl}$$

where $[(b^{(j)})]_{n_j}$ has the interpretation $\prod_{i=1}^{B^{(j)}} [b_i^{(j)}]_{n_j}$; similarly for $[(c^{(j)})]_{n_j}$.

Using (1.3) and (1.4) we observe that the above expression is equal to

$$x^{\alpha+km}\, k^m \sum_{l, n_1, \ldots, n_r = 0}^{\infty} \frac{(-px^s)^l}{l!} \left(\frac{\alpha + sl}{k} \right)_m \left(\frac{\alpha + sl}{k} + m \right)_{q_1 n_1 + \cdots + q_r n_r}$$

(*equation continued on p.* 151)

$$\times \left[1 - \sum_{j=1}^{r} u_j k\, x^k\, H_j(y_j)\right]^{-\left(\frac{\alpha + m + sl + k(q_1 n_1 + \ldots + q_r n_r)}{k}\right)}$$

$$\times \prod_{j=1}^{r} \frac{[(b^{(j)})]_{n_j}}{[(c^{(j)})]_{n_j}\, n_j!} \left[f_j(y_j)\,(u_j\, k\, x^k)^{q_j}\right]^{n_j}.$$

Now replacing u_j by u_j/kx^k $(j = 1, 2, \ldots, r)$ on both sides, we get our final result (2.1).

3. Applications

In this section, we shall apply our main theorem to obtain multilateral/multilinear generating functions for several well-known polynomial sets. We begin with the set of Hermite polynomials.

In view of the known relation [Srivastava and Singhal 1971, p. 76, eqn. (1.8)]

$$G_n^{(0)}(x, 2, 1, -1) = \frac{(-x)^n}{n!}\, H_n(x)$$

and the following formula due to Srivastava (1971, p. 4, eqn. (12))

$$\sum_{m_1, \ldots, m_n = 0}^{\infty} f(m_1 + \ldots + m_n) \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!} = \sum_{N=0}^{\infty} f(N) \frac{(x_1 + \ldots + x_n)^N}{N!}$$

we readily get from our main theorem the following.

Corollary 1—The classical Hermite polynomials satisfy the multilinear generating relation,

$$\sum_{n_1, \ldots, n_r = 0}^{\infty} H_{m+\Sigma n_i}(x) \prod_{j=1}^{r} H_{n_j}(y_j)\, u_j^{n_j}/n_j!$$

$$= \delta_r^{-(m+1)/2} \exp\left[x^2 - (\lambda_r^2 / \delta_r)\right] H_m(\lambda_r/\delta_r^{1/2}) \qquad \ldots(3.1)$$

where

$$\lambda_r = x - 2\sum_{j=1}^{r} u_j y_j,\ \delta_r = 1 - 4\sum_{j=1}^{r} u_j^2.$$

Formula (3.1) is due to Srivastava and Singhal [1972a, p. 1239, eqn. (4)].

The Konhauser biorthogonal polynomial sets $\{Y_n^{\alpha}(x; k)\}$ and $\{Z_n^{\alpha}(x; k)\}$ have the following *corrected* (and slightly improved) version of a multilateral generating function (of Patil and Thakare 1978) as already pointed out by Srivastava [1983, p. 370, eqn. (3)].

$$\textit{Corollary 2}— \sum_{n_1,\ldots,n_r=0}^{\infty} (m+\Sigma n_i)!\, Y^{\alpha}_{m+\Sigma n_i}(x;k) \prod_{j=1}^{r} Z^{\beta_j}_{n_j}(y_j;s)\, u_j^{n_j}/(1+\beta_j)_{sn_j}$$

$$= e^x \Delta_r^{-m-(1+\alpha)/k} \sum_{l=0}^{\infty} \frac{(-x\Delta_r^{-1/k})^l}{l!} \left(\frac{1+\alpha+l}{k}\right)_m$$

$$\times F^{1:0;\ldots;0}_{0:1;\ldots;1}\left(\begin{matrix}\left[m+\dfrac{1+\alpha+l}{k}:1,\ldots,1\right]:——;\dfrac{-y_1^s u_1}{\Delta_r},\ldots,\dfrac{-y_r^s u_r}{\Delta_r}\\ ——————:[1+\beta_1:s];\ldots;[1+\beta_r:s];\end{matrix}\right) \qquad \ldots(3.2)$$

where

$$\Delta_r = 1 - \sum_{i=1}^{r} u_j.$$

PROOF: For the Konhauser sets of biorthogonal polynomials $\{Y^{\alpha}_n(x;k)\}$ and $\{Z^{\alpha}_n(x;k)\}$ one has [Srivastava and Lavoie 1975, p. 315, eqn. (83))]

$$Y^{\alpha}_n(x;k) = k^{-n}\, G_n^{(\alpha+1)}(x,1,1,k),$$

and [Srivastava 1973, p. 490, Eqn. (7)]

$$\sum_{n=0}^{\infty} Z^{\alpha}_n(x;k)\, t^n/(1+\alpha)_{kn} = e^t\, {}_0F_k\,[—;(1+\alpha)/k,\ldots,(k+\alpha)/k;-(x/k)^k t].$$

Direct application of (2.1) yields (3.2).

The relation (3.2) contains the result (14) of Srivastava (1973, p. 491) and, for $k = s = 1$, it yields the result (3.3) below, which is due to Srivastava and Singhal [1972a, p. 1239, eqn. (5)]; see also Thakare (1980).

$$\textit{Corollary 3}— \sum_{n_1,\ldots,n_r=0}^{\infty} (m+\Sigma n_i)!\, L^{(\alpha)}_{m+\Sigma n_i}(x) \prod_{j=1}^{r} L^{(\beta_j)}_{n_j}(y_i)\, u_j^{n_j}/(1+\beta_j)_{n_j}$$

$$= (1+\alpha)_m\, e^x\, \Delta_r^{-\alpha-m-1}\, \Psi_2^{(r+1)}\Big[1+\alpha+m;\ 1+\alpha,\ 1+\beta_1,\ldots,\ 1+\beta_r;$$

$$-\frac{x}{\Delta_r},\ \frac{-y_1u_1}{\Delta_r},\ldots,\ \frac{-y_ru_r}{\Delta_r}\Big] \qquad \ldots(3.3)$$

where $\Delta_r = 1 - \sum_{j=1}^{r} u_j$ and $\Psi_2^{(n)}$ is the confluent hypergeometric function of n variables; see Exton (1976, p. 42).

Also from the known results,

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha}\, e^{xt} \text{ and } G_n^{\alpha}(x, 1, 1, -1) = L_n^{(\alpha-n)}(x)$$

we can obtain the generating relation

$$\textit{Corollary 4}— \sum_{n_1,\ldots,n_r=0}^{\infty} (m+\Sigma n_i)!\, L_{m+\Sigma n_i}^{(\alpha-(m+\Sigma n_i))}(x) \prod_{j=1}^{r} L_{n_j}^{(\beta_j-n_j)}(y_j)\, u_j^{n_j}$$

$$= (-1)^m\, e^x \{\mu_r\}^{\alpha-m} \sum_{l=0}^{\infty} \frac{(-x\,\mu_r)^l}{l!}\, (-\alpha-l)_m$$

$$\times F_A^{r*}(-\alpha-l-m; -\beta_1, \ldots, -\beta_r; u_1/\mu_r, \ldots, u_r/\mu_r) \qquad \ldots(3.4)$$

where

$$\mu_r = 1 - \sum_{j=1}^{r} u_j y_j,$$

$$F_A^{r*}(a; b_1, \ldots, b_r; x_1; \ldots; x_r) = \lim_{\delta\to\infty} F_A^r(a; b_1, \ldots, b_r; \delta, \ldots, \delta; \delta x_1, \ldots, \delta x_r),$$

and F_A^n is Lauricella's hypergeometric function of n variables of first kind; see Exton (1976, p. 41).

The following known formula

$$\sum_{n=0}^{\infty} n!\, L_n^{(\alpha-n)}(x)\, L_n^{(\beta-n)}(y)\, t^n = \alpha!\, t^{\alpha}\, e^{xyt}\, L_{\alpha}^{(\beta-\alpha)}\left(\frac{(1-xt)(1-yt)}{-t}\right) \qquad \ldots(3.5)$$

where α, β are non-negative integers, can be deduced from (3.4) by using the known relations,

$$\sum_{n=0}^{min(c,d)} \frac{(-c)_n\,(-d)_n}{n!}\, z^n = c!\, z^c\, L_c^{(d-c)}(-1/z); \text{ see Carlitz (1970, p. 166)}$$

and

$$\sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha}\, e^{-xt}\, L_m^{(\alpha)}(x\,(1+t))$$

see Carlitz (1960, p. 222).

For the Bessel polynomials of Krall and Frink, we can easily obtain a multilinear generating relation, by using (3.3) and the following relationship [see Srivastava *et al.* 1979, p. 15, eqn. (3.19)]

$$y_n(x, \alpha-2n, \beta) = (-x/\beta)^n\, n!\, L_n^{(1-\alpha)}(\beta/x), \quad (x \neq 0).$$

Our main theorem also yields as a particular case the next result.

Corollary 5—

$$\sum_{n_1,\cdots,n_r=0}^{\infty} (m+\Sigma n_i)!\, L_{m+\Sigma n_i}^{(\alpha)}(x) \prod_{j=1}^{r} H_{n_j}(y_j)\, u_j^{n_j}/n_j!$$

$$= (1+\alpha)_m\, e^x\, \{\Delta_r\}^{-1-\alpha-m}$$

$$\times F_{0:1;0}^{1:0;0}\left(\begin{matrix}[1+\alpha+m:1,2]: \text{————}; \\ \text{————}: [1+\alpha:1]; -;\end{matrix} -x/\Delta_r, -\sum_{j=0}^{r} u_j^2/\Delta_r^2\right)$$

where

$$\Delta_r = 1 - 2\sum_{j=1}^{r} u_j y_j.$$

Also we have the following multilateral generating function for the generalized Rice polynomials and the Laguerre polynomials.

*Corollary*6 —

$$\sum_{n_1,\ldots,n_r=0}^{\infty} (m+\Sigma n_i)!\, L_{m+\Sigma n_i}^{(\alpha)}(x) \prod_{j=1}^{r} H_{n_j}^{(\alpha_j,\beta_j-n_j)}(\xi_j,\eta_j,y_j)\, u_j^{n_j}/(1+\alpha_j)_{n_j}$$

$$= (1+\alpha)_m\, e^x\, \{\Delta_r\}^{-1-\alpha-m}$$

$$\times F_{0:1;2;\ldots;2}^{1:0;2;\ldots;2}\left(\begin{matrix}[1+\alpha+m:1,\ldots,1]: \text{————}: [(b^{(1)}):1];\ldots; \\ \text{————}: [1+\alpha:1]; [(c^{(1)}):1];\ldots; \\ [(b^{(r)}):1]; \\ [(c^{(r)}):1];\end{matrix} -x/\Delta_r, -y_1u_1/\Delta_r, \ldots, -y_ru_r/\Delta_r\right)$$

where

$$\Delta_r = 1 - \sum_{j=1}^{r} u_j, b_1^{(i)} = 1 + \alpha_i + \beta_i, b_2^{(i)} = \xi_i, c_1^{(i)} = 1 + \alpha_i,$$

$$c_2^{(i)} = \eta_i, (i = 1, 2, \ldots, r).$$

4. Other Applications

Srivastava (1969, p. 203) proved the following result,

$$\sum_{n=0}^{\infty} \frac{[f(x)\, z]^n}{n!}\, {}_{m+p}F_q\left[\begin{matrix}\Delta(m,-n), a_1,\ldots,a_p; \\ b_1,\ldots,b_q;\end{matrix} \quad (-m/f(x))^m \phi(x)\right]$$

$$= \exp[f(x)\, z]\, {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p; \\ b_1,\ldots,b_q;\end{matrix} \quad \phi(x)\, z^m\right]$$

where $f(x) \neq 0$, $\phi(x) \neq 0$, are real functions, $m = 1, 2, \ldots$, and $\Delta(\delta, \alpha)$ stands for the sequence of δ parameters $\frac{\alpha}{\delta}, \frac{\alpha+1}{\delta}, \ldots, \frac{\alpha+\delta-1}{\delta}$, $\delta \geqslant 1$.

Hence our theorem gives

$$\sum_{n_1,\ldots,n_r=0}^{\infty} (m+\Sigma n_i)!\; k^{-(m+\Sigma n_i)}\, G^{(\alpha)}_{m+\Sigma n_i}(x, s, p, k) \prod_{j=1}^{r} [f_j(y_j)\, u_j]^{n_j}/n_j!$$

$$\times {}_{m_j+B^{(j)}}F_{C^{(j)}}\left[\begin{matrix}\Delta(m_j,-n_j),(b^{(j)}); \\ (c^{(j)});\end{matrix} \quad (-m_j/f_j(y_j))^{m_j} \phi_j(y_j)\right]$$

$$= \exp(px^s)\, \Delta_r^{-m-(\alpha/k)} \sum_{l=0}^{\infty} \left(\frac{\alpha+sl}{k}\right)_m (-px^s\, \Delta_r^{-s/k})^l/l!.$$

$$\times F^{1 : B^{(1)};\ldots;B^{(r)}}_{0 : C(1);\ldots;C(r)}\left(\begin{matrix}[m+(\alpha+sl)/k : m_1,\ldots,m_r] : [(b^{(1)}) : 1];\ldots; \\ \text{---------------------} : [(c^{(1)}) : 1];\ldots; \\ [(b^{(r)} : 1]; \\ [(c^{(r)} : 1];\end{matrix} \quad \phi_1(y_1)\,(u_1/\Delta_r)^{m_1}, \ldots, \phi_r(y_r)\,(u_r/\Delta_r)^{m_r}\right) \qquad \ldots(4.1)$$

where

$$\Delta_r = 1 - \sum_{j=1}^{r} f_j(y_j)\, u_j.$$

Many interesting special cases, including most of the results in section 3 and results (5) to (11) in Brafman (1957), can be easily derived by giving particular values to the parameters; we however, omit the details.

For $x = 0$, $k = 1$, (4.1) reduces to

$$\sum_{n_1,\ldots,n_r=0}^{\infty} (\alpha)_{m+\Sigma n_j} \prod_{j=1}^{r} [f_j(y_j)\, u_j]^{n_j}/n_j!$$

$$\times {}_{m_j+B^{(j)}}F_{C^{(j)}}\left[\begin{matrix}\Delta(m_j,-n_j),\ (b^{(j)}); \\ (c^{(j)});\end{matrix}\quad (-m_j/f_j(y_j))^{m_j}\,\phi_j(y_j)\right]$$

$$= (\alpha)_m\,\Delta_r^{-\alpha-m}\, F^{1:\,B^{(1)};\ldots;\,B^{(r)}}_{0:\,C^{(1)};\ldots;C^{(r)}}\left(\begin{matrix}[\alpha+m : m_1,\ldots,m_r] : [(b^{(1)} : 1];\ldots; \\ \text{---------------} : [(c^{(1)}) : 1];\ldots; \\ [(b^{(r)}) : 1]; \\ [(c^{(r)}) : 1]:\end{matrix}\quad \phi_1(y_1)\,(u_1/\Delta_r)^{m_1},\ldots,\,\phi_r(y_r)\,(u_r/\Delta_r)^{m_r}\right) \qquad \ldots(4.2)$$

which contains the result of Srivastava (1969, p. 203).

For the associated Legendre functions $P_n^m(x)$, we have the generating function [Erdélyi *et al.* (1955, p. 266, eqn. (17))]

$$\sum_{n=0}^{\infty} P^m_{m+n}(x)\, t^n/(m+n)! = \exp(xt)\, 2^m\left(\Gamma\left(m+\frac{1}{2}\right)\right)^2(1-x^2)^{m/2}/\pi m!.$$

$$\times\ {}_2F_3\left[\begin{matrix}m+(1/2),\ m+(1/2); \\ 1/2,\ 1/2+m/2,\ 1+(m/2);\end{matrix}\quad -(1-x^2)\,t^2/4\right]$$

In view of this one can obtain from the relation (2.1) several generating functions involving the products of associated Legendre functions and classical polynomials.

For any arbitrary α, we mention the following generating function which is obtained in exactly the same way as (4.2).

$$\sum_{n=0}^{\infty} (\alpha)_n\, P^m_{m+n}(x)\, t^n/(m+n)! = (1-xt)^{-\alpha}\, 2^m\left(\Gamma\left(m+\frac{1}{2}\right)\right)^2(1-x^2)^{m/2}/\pi m!$$

(*equation continued on p.* 158)

$$\times {}_4F_3\left[\begin{matrix}\alpha/2,\ (1+\alpha)/2,\ m+(1/2),\ m+(1/2);\\ 1/2,\ (1+m),\ 1+(m/2);\end{matrix}\ (x^2-1)\,t^2/(1-xt)^2\right].$$

5. Remark

In our foregoing analysis we have refrained from listing multilateral generating functions involving several distinct functions. It needs to be emphasized that our main result subsumes several such mixed multilateral generating functions. As a sample we state here a multilateral generating function which involves seven known polynomials, namely the Laguerre polynomials, the generalized Rice polynomials, the Jacobi polynomials, the Bessel polynomials, the Charlier polynomials, the Gegenbauer polynomials and the Hermite polynomials.

$$\sum_{n_1,\ \ldots,n_6=0}^{\infty} (m+\Sigma n_i)!\, L^{(\alpha)}_{m+\Sigma n_i}(x)\,\frac{H^{(\alpha_1,\beta_1-n_1)}_{n_1}(\xi,\eta,y_1)\,P^{(\alpha_2,\beta_2-n_2)}_{n_2}(y_2)\,y_{n_3}(y_3,\alpha_3-n_3,\beta_3)}{(1+\alpha_1)_{n_1}(1+\alpha_2)_{n_2}\,n_3!}$$

$$\frac{C_{n_4}(\delta;y_4)\,C^{(\nu)}_{n_5}(y_5)\,H_{n_6}(y_6)}{(2\nu)_{n_5}\,n_4!\,n_6!}\ u_1^{n_1}\cdot u_2^{n_2}\cdots u_6^{n_6}$$

$$= \exp(x)\,(1+\alpha)_m\,\Delta_6^{-1-\alpha-m}$$

$$F\left(\begin{matrix}1:0;2;1;1;1;0;0\\ 0:1;2;1;0;0;1;0\end{matrix}\right)\Bigg(\begin{matrix}[1+\alpha+m:1,1,1,1,1,2,2]:\ \text{———};(b);\\ \text{———————————}:[1+\alpha:1];(c);\end{matrix}$$

$$\begin{matrix}[1+\alpha_2+\beta_2:1];\ [\alpha_3-1:1];\ [-\delta:1];\ \text{———};\ \text{———};\\ [1+\alpha_2:1];\ \ \text{———};\ \text{———};\ [\nu+1/2:1];\ \text{———};\end{matrix}\ -x/\Delta_6,$$

$$-y_1u_1/\Delta_6,\ (y_2-1)\,u_2/2\Delta_6,\ y_3u_3/\beta_3\Delta_6,\ u_4/y_4\Delta_6,\ (y_5^2-1)\,u_5^2/4\,\Delta_6^2,\ -u_6^2/\Delta_6^2\Bigg),$$

where $\Delta_6 = 1-[u_1+u_2+u_3+u_4+y_5u_5+2y_6u_6]$ and (b) denotes the two parameters, $b_1 = 1+\alpha_1+\beta_1$, $b_2 = \xi$. Similarly, (c) denotes the two parameters $c_1 = 1+\alpha_1$, $c_2 = \eta$.

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