Indian J. pure appl. Math., 15(1) : 15-22, January 1984

REGULAR REFLECTION OF A SHOCK WAVE FROM A RIGID WALL IN A STEADY PLANE FLOW OF A VIBRATIONALLY RELAXING GAS

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(*Received* 28 *April* 1983)

In a vibrationally relaxing gas the characteristic properties of the flow behind a reflected shock are discussed at the triple point (i.e. a point on the reflected shock at which the slip stream and the boundary curve meet). In the present paper, assuming the existence of a first order allowable discontinuity at the triple point, the jumps in the gradients of flow variables are obtained across the slip stream at this point. It is show that the flow in the region R_2 (i.e. the region between the slip stream and the boundary curve) will be irrotational at the triple point. Further, we conclude that the boundary curve introduces gradients, as a consequence of which the stream lines become curved in the region R_2 at the triple point. Some other interesting features of the flow behind the reflected shock are discussed. Finally the vorticity jump across the slip stream is discussed.

1. INTRODUCTION

For a plane incident shock wave at the surface of a rigid wall, the regular reflection or Mach reflection occurs depending upon the angle of incidence. In a regular reflection, there is a triple point T on the reflected shock at which the curvature of the shock first begins. In Mach reflection the three shocks, incident, reflected and Mach, meet at the triple point. The Mach reflection has been discussed by Sternberg (1959) and Smith (1959). Boublil and Earnshaw (1964) have given theoretical and experimental results of the magnetohydrodynamic interactions between a uniform transverse magnetic field and a reflected shock wave. Spence (1961) was the first to study analytically the unsteady flow field behind a reflected shock wave in a relaxing gas. Baganoff (1965) measured the end-wall pressure histories in shock reflection processes. The method of characteristics has been employed to calculate the reflected shock wave flow field in the presence of relaxation; Johannesen, Bird and Zienkiewicz (1967) combined the Rankine—Hugoniot, characteristic, and Rayleigh-line equations, and made calculations for vibrational relaxation in carbon dioxide. Presley and Hanson (1969) also employed the method of characteristics and carefully calculated the time-dependent reflected shock-wave flow field with

chemical reaction. Later, Hanson (1971 a, b) used this concept to study vibrational relaxation in carbon monoxide. Brandon (1969) calculated shock wave reflection with vibrational relaxation behind the incident shock wave and simultaneous chemical and vibrational relaxation behind the reflected shock wave. Buggisch (1969) delineated the analytical solution for the case when the shock is weak but vibrational effects are strong relative to the shock strength; Buggisch (1970) considered the case of a relatively strong shock perturbed by weak vibrational effects, whereas Hung and Seebass (1974) considered the case of weak shock when the effects of vibrational relaxation are of the same order as the shock strength.

In this paper we aim to study the flow behind a reflected shock at a point where the slip stream and the boundary curve meet the reflected shock. In this context, the work carried out by Taub (1953) and Pant (1968) is also worth mentioning.

Here we have considered only the regular reflection. The region behind the reflected shock OTB is divided by the slip-stream TS, emanating from the triple point T, into two regions, i.e. the region behind the curved portion TB and the region behind the straight portion OT of the reflected shock. The state upstream from the incident shock is assumed to be uniform. Without any loss of generality, the flow variables can be assumed constant in the region IOTB, since the incident shock is a straight shock. The reflected shock will be curved

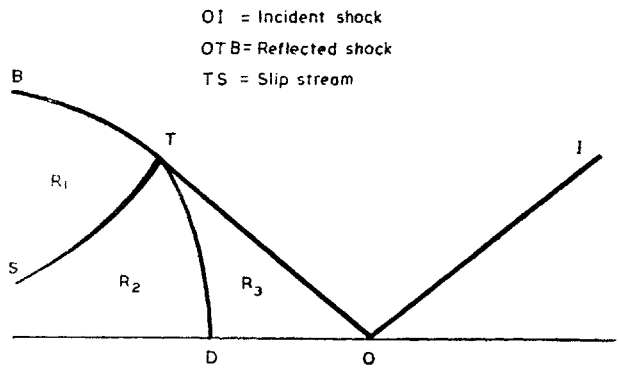

FIG.. 1. Oblique regular Reflection of a plane shock from a Rigid Wall.

beyond the point T due to the influence of sound signal. At T, the curvature of the shock is discontinuous, i. e. the straight portion of the shock is tangent to the curved portion TB. Since the normal mass flux across a slip stream is zero, we have $[p] = 0$, where the bracket [ ] denotes the jump in the quantity enclosed across the discontinuity surface and p is the pressure of the vibrationally relaxing gas.

The region below TS is uniform due to the straight portion of the shock, whereas the region above TS is not uniform due to the curved portion TB of the shock. Hence we assume that a first order allowable discontinuity exists at T. Thus, the following conditions hold at T :

(i) The pressure p is continuous across TS.

(ii) The curvature of the stream line, just above TS, at T is continuous.

A stream line above TS has a finite curvature but below TS has no curvature. Thus the condition (ii) is violated. To avoid this difficulty Taub (1953) has shown that if the flow variables behind the straight portion of the shock are assumed to vary in a particular manner along a boundary curve, then the condition (ii) is satisfied. The boundary curve TD separates the region R_3 of uniform state from the region R_2 of continuously varying state. The region betweeen the curved portion of the reflected shock and the slip stream is denoted by R_1.

2. Basic Equations

The continuity, momentum and energy equations, for the steady flow of a vibrationally relaxing gas, are

$$u_i \rho_{,i} + \rho u_{i,i} = 0 \qquad \ldots(1)$$

$$\rho u_j u_{i,j} + p_{,i} = 0 \qquad \ldots(2)$$

$$\rho u_i h_{,i} - u_i p_{,i} = 0 \qquad \ldots(3)$$

where ρ, p, u_i and h are respectively the density, pressure, velocity components and enthalpy of the gas. The summation convention on repeated indices is employed, and a comma followed by an index i denotes a partial derivative with respect to space variable x_i. Throughout this paper we shall take the spectrum of suffices i, j, k to be 1, 2

The equations of state, when the gas is capable of being excited in its vibrational mode of internal energy, are

$$p = \rho RT \qquad \ldots(4)$$

$$h = \frac{\gamma}{(\gamma - 1)} \frac{p}{\rho} + e_v \qquad \ldots(5)$$

where R, T and γ denote the gas constant, the temperature and the frozen specific heat ratio respectively, and e_v is the vibrational energy per unit mass given by

$$u_i e_{v,i} = \omega \{e_v^* (T) - e_v\} \equiv \chi \qquad \ldots(6)$$

where $e_v^* (T)$ is the equilibrium value of the vibrational energy at the local temperature given by

$$e_v^* (T) = R \theta_v / \{\exp (\theta_v / T) - 1\}$$

where θ_v is the characteristic temperature of the molecular vibration. The quantity ω in (6) is the relaxation frequency given by Bray (1970, p. 67)

$$\omega = Ap \exp (-BT^{-1/3})$$

where A and B are positive constants depending on the physical properties of the gas only.

Equation (3) with the help of (1), (5) and (6) is conveniently transformed into

$$u_i\, p_{,i} + \rho\, a_f^2\, u_{i,i} + \Lambda = 0 \qquad \ldots(7)$$

where $\Lambda = \rho\,(\gamma - 1)\,\chi$ and a_f is the frozen speed of sound given by $a_f^2 = \gamma p/\rho$.

3. Flow Behind the Reflected Shock

The plane shock configuration in a steady flow may be represented by $x_i = x_i(s)$, where x_i $(i = 1,2)$ are the rectangular cartesian coordinates of a point on the shock and the parameter s measures the arcual length along the shock curve. Any stream line crossing the reflected shock may be represented by the curve $s =$ constant. If τ be the parameter of arc length along a stream line measured from the shock curve, then every flow variable in the region behind the reflected shock is a function of s and τ. Thus we have

$$f_{,i} = s_{,i}\, \partial_s f + \tau_{,i}\, \partial_\tau f \qquad \ldots(8)$$

where $f(s, \tau)$ denotes any of the flow variables. ∂_s and ∂_τ are the operators of partial differentiation with respect to s and τ respectively. Let λ_i and n_i be respectively the components of unit tangent and unit normal vectors to the curve $\tau =$ constant. The shock curve OTB coincides with the curve $\tau = 0$. Thus, we have the following relations given by Taub (1953) :

$$\left.\begin{aligned} &\partial_s x_i = \lambda_i,\ \partial_\tau x_i = u_i/q,\ q^2 = u_i\, u_i \\ &n_i = e_{ij}\, \lambda_j,\ \lambda_i = -\, e_{ij}\, n_j \\ &s_{,i} = -\,(1/u_n)\, e_{ij}\, u_j,\ \tau_{,i} = (q/u_n)\, n_i \end{aligned}\right\} \qquad \ldots(9)$$

where $e_{11} = e_{22} = 0$, $e_{12} = -\, e_{21} = 1$, $u_n = u_i n_i$ and $u_t = u_i\, \lambda_i$. Using (8) and (9), eqns. (1), (2) and (7) can be transformed into the following forms

$$u_n\, q\, \partial_\tau \rho + \rho q n_i\, \partial_\tau u_i - \rho\, e_{ij}\, u_j\, \partial_s u_i = 0 \qquad \ldots(10)$$

$$\rho\, u_n\, q\, \partial_\tau u_i - e_{ij}\, u_j\, \partial_s p + q n_i\, \partial_\tau p = 0 \qquad \ldots(11)$$

$$q\, \partial_\tau p - q\, a_f^2\, \partial_\tau \rho + \Lambda = 0. \qquad \ldots(12)$$

Theorem 1 — The following relations, between the variations of flow parameters along a stream line and along the curve $\tau =$ constant, hold :

$$\partial_\tau \rho = (q a_f^2)^{-1}\, (\rho\, u_n\, e_{ij}\, u_j\, \partial_s u_i - u_t\, \partial_s p - \Lambda)\, (M_f^2 - 1)^{-1} \qquad \ldots(13)$$

$$\partial_\tau p = q^{-1}\, (\rho\, u_n\, e_{ij}\, u_j\, \partial_s u_i - u_t\, \partial_s p - \Lambda M_f^2)\, (M_f^2 - 1)^{-1} \qquad \ldots(14)$$

$$\partial_\tau u_i = (\rho\, u_n)^{-1}\, (q^{-1}\, e_{ij}\, u_j\, \partial_s p - n_i\, \partial_\tau p) \qquad \ldots(15)$$

where $M_f = (u_n/a_f)$ is the forzen Mach number.

Proof : Multiplying (11) by n_i and eliminating $n_i\, \partial_\tau u_i$ and $\partial_\tau p$ with the help of (10) and (12), we get (13). Substituting from (13) in (12) and (11), we get the relations (14) and (15).

Theorem 2 — The gradients of flow variables along the boundary curve *TD*, separating the regions R_2 and R_3, are given by

$$u_{i,j} = \xi\, \bar{n}_i\, \bar{n}_j \qquad \ldots(16)$$

$$p_{,i} = -\rho\xi \bar{u}_n\, \bar{n}_i \qquad \ldots(17)$$

$$\rho_{,i} = -(\rho\xi/\bar{u}_n)\, \bar{n}_i \qquad \ldots(18)$$

where $\xi = \Lambda / \rho\,(\bar{u}_n^2 - a_f^2)$, $\bar{u}_n = u_i\, \bar{n}_i$ and $\bar{n}_i$ are the components of the unit normal vector to the boundary curve TD.

PROOF : The portion OT of the reflected shock is of constant strength and therefore will not introduce gradients to flow variables in region R_3. Hence the flow variables u_i, p and ρ are constant along the boundary curve. Thus we have

$$u_{i,j}\, \bar{\lambda}_j = 0,\ p_{,j}\, \bar{\lambda}_j = 0,\ \rho_{,j}\, \bar{\lambda}_j = 0 \qquad \ldots(19)$$

where $\bar{\lambda}_j$ are the components of unit tangent vector to the boundary curve.

From (19) we can write

$$u_{i,j} = \alpha_i\, \bar{n}_j,\ \rho_{,i} = \zeta\, \bar{n}_i. \qquad \ldots(20)$$

Multiplying (2) by $\bar{\lambda}_i$ and using (19) and (20), we get

$$\rho\, \bar{u}_n\, \alpha_i\, \bar{\lambda}_i = 0. \qquad \ldots(21)$$

We take the boundary curve such that $\bar{u}_n \neq 0$. Thus, from (21) we have

$$\alpha_i\, \bar{\lambda}_i = 0 \Rightarrow \alpha_i = \xi \bar{n}_i \Rightarrow u_{i,i} = \xi. \qquad \ldots(22)$$

Eliminating $u_i\, p_{,i}$ between (2) and (7) and using (20) and (22), we get

$$\xi = \Lambda/\rho\,(\bar{u}_n^2 - a_f^2). \qquad \ldots(23)$$

Using (22) in (20), we get (16). Using (16) in (2) we get (17) and finally substituting from (22) and (20) in (1) we get (18).

Incidently, from (16), we conclude that the vorticity in the region R_2 at the triple point T is zero. Thus, in view of the relations (16) to (18), we see that the boundary curve introduces only gradients, as a consequence of which the curvature is introduced to the stream lines in the region R_2 at the triple point T.

Theorem 3 — The following relations hold between the quantities of regions R_1 and R_2 :

$$u_t\,(\partial_\tau\, p)_{R_1} - q\,(\partial_s\, p)_{R_1} = -\rho\xi u_n\, q^{-1}\, \bar{u}_n\, \bar{u}_t \qquad \ldots(24)$$

$$(\partial_\tau\, \rho)_{R_1} = (\Lambda - \rho\, \xi\, \bar{u}_n^2) / q\, a_f^2 \qquad \ldots(25)$$

where $\bar{u}_t = u_i\, \bar{\lambda}_i$.

PROOF : The curvature K of a stream line is given by $q^3 K = -e_{ij}\, u_j\, u_k\, u_{i,k}$. Substituting for $u_k\, u_{i,k}$ from (2) and taking jump across TS at T we get $\rho\, q^3\, [K] = e_{ij}\, u_j\, [p,_i]$. Since the curvature of a stream line is continuous across the slip stream *TS* at *T*, we have

$$e_{ij}\, u_j\, (p,i)_{R_1} = e_{ij}\, u_j\, (p,i)_{R_2}. \qquad \ldots(26)$$

By virtue of (8) and (9), we have

$$e_{ij}\, u_j\, (p,i)_{R_1} = (q/u_n)\, \{u_t\, (\partial_\tau\, p)_{R_1} - q\, (\partial_s\, p)_{R_1}\} \qquad \ldots(27)$$

and from (17), we have

$$e_{ij}\, u_j\, (p,i)_{R_2} = -\, \rho \xi \bar{u}_n\, \bar{u}_t. \qquad \ldots(28)$$

Substituting from (27) and (28) in (26) we get (24). The value of $(\partial_\tau\, p)_{R_1}$ in (24) is determined from (14). The variations of flow parameters along the curved portion of the reflected shock can be expressed in terms of the strength and curvature of the shock and the upstream flow conditions (Ram 1968).

Since the variation of pressure along the slip stream is continuons, we have

$$u_i\, (p,i)_{R_1} = u_i\, (p,i)_{R_2}.$$

From (17), we have

$$u_i\, (p,i)_{R_2} = -\, \rho\, \xi\, \bar{u}_n^2 \qquad \ldots(29)$$

and from (8), (9) and (12), we have

$$u_i\, (p,i)_{R_1} = q\, a_f^2\, (\partial_\tau\, \rho)_{R_1} - \Lambda. \qquad \ldots(30)$$

Substituting from (29) and (30) in (28), we get (25).

Theorem 4 — The jumps in the gradients of flow variables across the slip stream at T are given by

$$[u_{i,j}] = q^{-2}\, \{(1/\rho u_n)\, (e_{ik}\, u_k - (u_t/u_n)\, u_i)\, (\partial_s\, p)_{R_1} + \xi \bar{u}_n\, (n_i - \bar{n}_i)\}\, u_j$$

$$+\, q^{-2} \{\xi\, (\bar{u}_n\, n_i - \bar{u}_t\, \bar{n}_i) - (q^2/u_n)\, (\partial_s\, u_i)_{R_1}\}\, e_{jk}\, u_k \qquad \ldots(31)$$

$$[p,i] = \rho\, \xi\, (a_f^2 - 1)\, (\bar{u}_n/q\, a_f)^2\, u_i - q^{-2}\, \{(q^2/u_n)\, (\partial_s p)_{R_1}$$

$$+\, \rho\, \xi\, (u_t\, \bar{u}_n^2 / u_n) - \rho \xi\, \bar{u}_t\, \bar{u}_n\}\, e_{ij}\, u_j \qquad \ldots(32)$$

$$[\rho,i] = (q a_f)^{-2}\, \{\Lambda + \rho \xi\, (a_f^2 - \bar{u}_n^2)\}\, u_i + q^{-2}\, \{(u_t/u_n\, a_f^2)\, (\Lambda - \rho \xi\, \bar{u}_n^2)$$

$$+\, (\rho \xi\, \bar{u}_t/\bar{u}_n) - (q^2/u_n)\, (\partial_s\, \rho)_{R_1}\}\, e_{ij}\, u_j. \qquad \ldots(33)$$

PROOF : Since the flow variables are continuous across the slip stream at T, we have

$$[u_{i,j}] = [\gamma_{ii}]\, u_j + [\nu_i]\, e_{jk}\, u_k \qquad \ldots(34)$$

where

$$[\gamma_{ii}] = q^{-2}\, u_j\, \{(u_{i,j})_{R_1} - (u_{i,j})_{R_2}\}$$

and

$$[\nu_i] = q^{-2}\, e_{jk}\, u_k\, \{(u_{i,j})_{R_1} - (u_{i,j})_{R_2}\}.$$

Using (16), we have

$$u_j\,(\,u_{i,j}\,)_{R_2} = \xi\,\bar{u}_n\,\bar{n}_i,\; e_{jk}\,u_k\,(\,u_{i,j}\,)_{R_2} = \xi\,\bar{u}_t\,\bar{n}_i. \qquad \ldots(35)$$

Using (8), (9), (11), (12) and (25), we have

$$u_j\,(\,u_{i,j}\,)_{R_1} = \xi\bar{u}_n\,n_i + (\rho\,u_n)^{-1}\,e_{ij}\,u_j\,(\partial_s\,p)_{R_1} \qquad \ldots(36)$$

and

$$e_{jk}\,u_k\,(u_{i,j})_{R_1} = -\,(q^2/u_n)\,(\partial_s\,u_i)_{R_1} + (u_i/u_n)\,\{\xi\,\bar{u}_n^2\,n_i + (\rho\,u_n)^{-1}\,e_{ij}\,u_j\,(\partial_s\,p)_{R_1}\}. \qquad \ldots(37)$$

Substituting from (35) to (37) in (34), we get (31).

To obtain jump in the pressure gradient, we take

$$[\,p,i] = [\eta]\,u_i + [\nu]\,e_{ij}\,u_j \qquad \ldots(38)$$

where

$$[\eta] = (u_i/q^2)\,\{(p,i)_{R_1} - (\,p,i)_{R_2}\}$$

and

$$[\nu] = q^{-2}\,e_{ij}\,u_j\,\{\,(p,i)_{R_1} - (\,p,i)_{R_2}\}.$$

By virtue of (17), we have

$$u_i\,(p,i)_{R_2} = -\,\rho\xi\bar{u}_n^2,\quad e_{ij}\,u_j\,(p,i)_{R_2} = -\,\rho\xi\,\bar{u}_t\,\bar{u}_n. \qquad \ldots(39)$$

Using (8), (9), (12) and (25), we have

$$u_i\,(\,p,i)_{R_1} = -\,\rho\,\xi\,\bar{u}_n^2 \qquad \ldots(40)$$

and

$$e_{ij}\,u_j\,(\,p,i)_{R_1} = (\rho u_t/u_n)\,\xi\,\bar{u}_n^2 - (q^2/u_n)\,(\partial_s\,p)_{R_1}. \qquad \ldots(41)$$

Substituting from (39) to (41) in (38), we get (32).

Similarly proceeding for the density gradients, we obtain

$$[\eta] = (q\,a_f)^{-1}\,\{\Lambda + \rho\xi\,(a_f^2 - \bar{u}_n^2)\} \qquad \ldots(42)$$

and

$$[\nu] = q^{-2}\,\{(u_t/u_n\,a_f^2)\,(\Lambda - \rho\xi\,\bar{u}_n^2) + (\rho\xi\,\bar{u}_t/\bar{u}_n) - (q^2/u_n)\,(\partial_s\,\rho)_{R_1}\}. \qquad \ldots(43)$$

Relation (33) follows from (42) and (43).

Theorem 5 — The jump in the vorticity ω across the slip stream at T is given by

$$[\omega] = \xi\,\bar{u}_n\,q^{-2}\,(u_n - u_t) - (\rho\,u_n)^{-1}\,\{u_i\,(\partial_s\,u_i)_{R_1} + (\,\partial_s\,p\,)_{R_1}\}. \qquad \ldots(44)$$

PROOF : The vorticity in a two-dimensional flow is given by the relation

$$\omega = \tfrac{1}{2}\,(u_{j,i} - u_{i,j})\,e_{ij}. \qquad \ldots(45)$$

Taking jump in (45) across the slip stream at T and using (31), we get (44).

ACKNOWLEDGEMENT

One of us (R. S.) is thankful to the University Grants Commission for the award of a Research Associateship.

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