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A TOPOLOGICAL RING OF FUNCTIONS AND A UNIFORM RING OF FUNCTIONS

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The main aim of this paper is the constructions of the topological ring m_{J_S} (E, F) and the uniform ring $m_{\mathcal{U}_S}$ (E, F). We also obtain distinction between a topological ring and a uniform ring. In fact we have proved that every uniform ring is a topological ring but the converse is not, true. In the construction of m_{J_S} (E, F), E is a non-empty set, F a topological ring, m (E, F) the set of all maps from E to F and S a suitable family of subsets of E, where as in the construction of $m_{\mathcal{U}_S}$ (E, F), E is a non-empty set, F a uniform ring, m (E, F) the set of all maps from E to F and S a suitable family of subsets of E satisfying a weaker condition than the condition satisfied by S in the construction of m_{J_S} (E, F). The construction of m_{J_S} (E, F) being an analogue of Theorem 1 of Jha (1973) on construction of topological group m_{J_S} (E, F).

INTRODUCTION

The purpose of the present paper is the construction of the topological ring of functions m_{J_S} (E, F) and the construction of the uniform ring of functions $m_{\mathcal{U}_S}$ (E, F). In the construction of m_{J_S} (E, F), E is a non-empty set, F a topological ring, m (E, F) the set of all maps from E to F and S a suitable family of subsets of E, whereas in the construction of $m_{\mathcal{U}_S}$ (E, F). E is a non-empty set, F a uniform ring, m (E, F) the set of all maps from E to F and S a family of subsets of E satisfying a weaker condition than the condition satisfied by S in the construction of m_{J_S} (E, F). We have explored two special cases of uniformity $\mathcal{U}_S$ on the uniform ring

$m_{\mathcal{U}_S}(E, F)$. If S is the set of all finite subsets of E, this uniformity is that pointwise convergence and if $S = \{E\}$ the uniformity is that of uniform convergence [see Kelley (1955, pp. 220, 226)].

In fact we introduce the notion of uniform ring and prove that every uniform ring is a topological ring but the converse is not true.

The construction of $m_{J_S}(E, F)$ is an analogue of Theorem 1 of Jha (1973) on the construction of topological group $m_{J_S}(E, F)$.

We start with the following definition (see Hussain 1966, Kowalsky 1965).

Definition 1—Let (E, T) be a topological ring. Let $\{U\}$ be a fundamental system of neighbourhoods of 0 (the zero element in E). Let $M \subseteq E$. Then M is said to be left (right) bounded if $\forall\ U \in \{U\}$, $\exists\ V \in \{U\}$ such that $V \cdot M \subseteq U$ $(M \cdot V \cdot \subseteq U)$. M will be called bounded if it is both left bounded and right bounded [see Kowalsky (1965, Chap. 7, p. 255)].

*Theorem**—In each topological ring E, there exists a fundamental system $\{U\}$ of closed neighbourhoods of zero element 0, such that

(i) Each U is symmetric;

(ii) For each $U \in \{U\}$ there exists $V \in \{U\}$ such that $V + V \subseteq U$;

(iii) For each $U \in \{U\}$ there exists $V \in \{U\}$ such that $V \cdot V \subseteq U$;

(iv) For each $U \in \{U\}$ and $a \in E$ there exists $V \in \{U\}$ such that $a \cdot V \subseteq U$ and $V \cdot a \subseteq U$.

Conversely, given a ring E with a filter base $\{U\}$ satisfying (i)–(iv) then there exists a unique topology T on E such that (E, T) is a topological ring and $\{U\}$ forms a fundamental system of neighbourhoods of zero element 0. (see Theorem 3 of Hussain (1966, p. 46) and Theorem 38.1, of Kowalsky (1965, p. 254)].

Main Results

Theorem 1— Let E be a set. Let F be a topological ring. Let $m(E, F)$ be the set of all maps from E to F. Let S be a family of subsets of E satisfying the following conditions :

(i) $A, B \in S \Rightarrow \exists\ C \in S$ such that $A \cup B \subseteq C$.

(ii) $f(A)$ is bounded in F for each $A \in S$ and $f \in m(E, F)$.

Then there exists a ring structure on $m(E, F)$ and a unique topology J_S compatible with this ring structure such that $m_{J_S}(E, F)$ is a topological ring.

Proof : We first note that by the usual definition of addition and multiplication of maps given by

$$(f + g)(x) = f(x) + g(x)$$

and

$$(f \cdot g)(x) = f(x) \cdot g(x), \text{ for all } x \in E$$

the set $m(E, F)$ can be given a ring structure. If 0 is the zero element of F then the map f_0 given by $f_0(x) = 0$ for all $x \in E$, serves as the zero element of $m(E,F)$. We now propose to define a compatible topology on the ring $m(E, F)$. Since F is a topological ring, there exists a fundamental system $\{U\}$ of closed neighbourhoods of the zero element 0 possessing properties I–IV of theorem*.

For each $A \in S$ and $U \in \{U\}$ we define

$$[A, U]_m = \{f; f \in m(E, F) \text{ and } f(A) \subseteq U\}.$$

We claim that the family $\{[A, U_m]\}$ where A varies over S and U varies over $\{U\}$ forms a fundamental system of neighbourhoods of the zero element f_0 in $m(E, F)$ such that $m(E, F)$ is a topological ring.

To verify our claim let $[A, U]_m$ and $[B, V]_m$ be any two members in the family $\{[A, U]_m\}$. Then (see Jha 1973) $\exists\ [C, W]_m \in \{[A, U]_m\}\ A \in S,\ U \in \{U\}$ such that $[C, W]_m \subseteq [A, U]_m \cap [B, V]_m$. Thus it follows that the family $\{[A, U]_m\}$ is a filter base on $m(E, F)$. Secondly we note that the symmetry of U implies that each $[A, U]_m$ is symmetric (Jha 1973).

Next consider $[B, V]_m \in \{[A, U]_m\}$ then $\exists\ W \in \{U\}$ such that $[B, W]_m + [B, W]_m \subseteq [B, V]_m$ (see Jha 1973).

Again if $[B, V]_m \in \{[A, U]_m\}$ then on account of (c) $\exists\ W \in \{U\}$ such that $W \cdot W \subseteq V$ and it will follow that $[B, W]_m$. $[B, W]_m \subseteq [B, V]_m$. Finally consider any $[B, V]_m \in \{[A, U]_m\}$ and $f \in m(E, F)$. Since $f(B)$ is bounded $\exists\ W \in \{U\}$ such that $W \cdot f(B) \subseteq V$ and $f(B) \cdot W \subseteq V$. It follows, then that $f \cdot [B, W]_m \subseteq [B \cdot V]_m$ and $[B, W]_m \cdot f \subseteq [B, V]_m$. In fact if $h \in [B, W]_m$ then $h(B) \subseteq W \cdot$ and so $(f \cdot h)(B) \subseteq f(B) \cdot h(B)$

$$\subseteq f(B) \cdot W$$

$$\subseteq V.$$

Therefore,

$$f \cdot [B, W]_m \subseteq [B, V]_m.$$

Similarly,

$$[B, W]_m \cdot f \subseteq [B, V]_m.$$

Hence by Theorem* there exists a unique topology J_S on $m(E, F)$ such that $m_{J_S}(E, F)$ is a topological ring and the family $\{[A, U]_m\}$ forms a fundamental system of neighbourhoods of the zero element f_0 in $m(E, F)$. This completes the proof.

Definition 2—Let E be a ring which is also a uniform space. Let $\mathcal{B}$ be a base for the uniformity $\mathcal{U}$ of E. Let $U \in \mathcal{B}$. Then

(i) U is called translation-invariant under addition if $z \in E$ and $(x, y) \in U$ $\Rightarrow (z + x, z + y) \in U$; $+$ is commutative in E

(ii) U is called translation-invariant under multiplication if $z \in E$ and $(x, y) \in U \Rightarrow (z \cdot x, z \cdot y) \in U$ and $(x \cdot z, y \cdot z) \in U$.

The base $\mathcal{B}$ is called translation-invariant if each $U \in \mathcal{B}$ is translation invariant under both addition and multiplication.

Definition 3— By a uniform ring we shall mean a ring with a uniformity having translation-invariant base (under addition and multiplication).

Lemma— If E is a (multiplicative) Semi group and $V \subseteq E \times E$ is translation-invariant under multiplication, then

$$V \cdot V \subseteq V^2.$$

PROOF : Let $(x_1, y_1) \in V$ and $(x_2, y_2) \in V$.

Now since V is translation-invariant under multiplication we have $(x_1 \cdot x_2, y_1 \cdot x_2) \in V$ and $(y_1 \cdot x_2, y_1 \cdot y_2) \in V$ which imply that $(x_1 \cdot x_2, y_1 \cdot y_2) \in V^2$. Thus

$$V \cdot V \subseteq V^2$$

This completes the proof.

Theorem 2— Every uniform ring is a topological ring, but the converse is not true.

PROOF : Let $(E, \mathcal{U})$ be a uniform ring and $\mathcal{B}$ a translation-invariant base for the uniformity $\mathcal{U}$. To prove that E is topological ring we must show that the mappings : $(x, y) \to x + y$, $x \to -x$ and $(x, y) \to x \cdot y$ are continuous. For this let $U[x + y]$, $U[-x]$ and $U[x \cdot y]$ be any basic neighbourhoods of $x + y$, $-x$ and $x \cdot y$ respectively. Since $U \in \mathcal{B}$ then there exists $W, V \in \mathcal{B}$ such that $V^2 \subseteq U$ and $W \subseteq U^{-1}$. Then it is easy to verify that

$$V[x] + V[y] \subseteq U[x + y], \; -W[x] \subseteq U[-x]$$

and

$$V[x] \cdot V[y] \subseteq U[x \cdot y].$$

Now we shall verify only that $V[x] \cdot V[y] \subseteq U[x \cdot y]$.

Let $z_1 \in V[x]$ and $z_2 \in V[y]$. Then

$$z_1 \in V[x] \Rightarrow (x, z_1) \in V \text{ and } z_2 \in V[y] \Rightarrow (y, z_2) \in V.$$

This implies $(x \cdot y, z_1 \cdot z_2) \in V \cdot V \subseteq V^2 \subseteq U$ (by lemma).

And so $z_1 \cdot z_2 \in U[x \cdot y]$.

Thus $V[x] \cdot V[y] \subseteq U[x \cdot y]$, and so E is topological ring.

To prove that the converse is not true, consider the following example :

Let $E = R$ the set of all real number. Then R is a topological ring under the usual operations of addition and multiplication with the usual metric topology defined by

$$d(x, y) = | x - y | .$$

Let

$$W_n = \left] -\frac{1}{n}, \frac{1}{n} \right[, \; n = 1, 2, 3 \ldots$$

then we know that $\left\{ W_n \right\}_{n=1}^{\infty}$ is a fundamental system of neighbourhoods of the zero element in R.

For W_n in $\left\{ W_n \right\}_{n=1}^{\infty}$ we define

$$L(W_n) = \{(x, y) \in R \times R : -x + y \in W_n\}.$$

Then we know the family $\{L(W_n)\}$ where W_n runs over $\{W_n\}$ forms a base $\mathscr{B}$ for a uniformity $\mathscr{U}$ on R. (See Hussain 1966, p. 52). Clearly each member of $\{L(W_n)\}$ is translation-invariant under addition. Now we prove that $\{L(W_n)\}$ is not translation-invariant under multiplication.

Let

$$L(W_n) \in \{L(W_n)\}, \; n > 2.$$

Now

$$L(W_n) = \{(x, y) : -x + y \in W_n\}$$
$$= \left\{ (x, y) : -\frac{1}{n} < -x + y < \frac{1}{n} \right\}.$$

Then

$$\left(\frac{1}{2(n-1)}, \frac{1}{(n-1)} \right) \in L(W_n). \text{ Let } (n+1) \in R.$$

But

$$\left((n+1)\cdot\frac{1}{2(n-1)},\ (n+1)\cdot\frac{1}{(n-1)}\right) \notin L(W_n).$$

Therefore $L(W_n)$ is not translation-invariant under multiplication. Thus the base $\mathscr{B} = \{L(W_n)\}$ is not translation-invariant under multiplication. Let $\mathscr{B}_1$ be any base for the uniformity $\mathscr{U}$ on R. Since $\mathscr{B}$ and $\mathscr{B}_1$ are bases for the uniformity $\mathscr{U}$ on R. So for $L(W_n) \in \mathscr{B}$ there exists $V \in \mathscr{B}_1$ such that $V \subseteq L(W_n)$. Also since $V \in \mathscr{B}_1$ so there exists $L(W_k) \in \mathscr{B}$ such that $L(W_k) \subseteq V$. Thus $L(W_k) \subseteq V \subseteq L(W_n)$. Then it is easy to see that $k \geqslant n$. Closely $\left(\frac{1}{k+1}, \frac{2}{k+1}\right) \in L(W_k)$ then $\left(\frac{1}{k+1}, \frac{2}{k+1}\right) \in V$. Now we shall prove $\left(\frac{a}{k+1}, \frac{2a}{k+1}\right) \notin V$ for some $a \in R$. Since $V \subseteq L(W_n)$, $\left(\frac{1}{k+1}, \frac{2}{k+1}\right) \in L(W_n)$.

Let

$$a = \frac{k+1}{n} + 1. \text{ Since } \frac{-a}{k+1} + \frac{2a}{k+1} = \frac{1}{n} + \frac{1}{k+1} > \frac{1}{n}$$

then

$$\left(\frac{a}{k+1}, \frac{2a}{k+1}\right) \notin L(W_n) \Rightarrow \left(\frac{a}{k+1}, \frac{2a}{k+1}\right) \notin V.$$

Thus $\mathscr{B}_1$ is not a translation-invariant base. Hence R is not uniform ring.

Theorem 3— Let E be a non-empty set and S a family of subsets of E satisfying the condition, $A, B \in S \Rightarrow \exists\, C \in S$ such that $A \cup B \subseteq C$. Let F be a Hausdorff uniform ring. Let $m(E, F)$ be the set of all maps of E into F. Then there exists a ring structure on $m(E, F)$ and a translation-invariant base for a unique uniformity $\mathscr{U}_S$ for $m(E, F)$ such that $m_{\mathscr{U}_S}(E, F)$ is an uniform ring.

PROOF: We first note that by the usual definition of addition and multiplication of maps given by

$$(f+g)(x) = f(x) + g(x)$$

and

$$(f \cdot g)(x) = f(x) \cdot g(x), \text{ for all } x \in E$$

the set $m(E, F)$ can be given a ring structure. If 0 be the zero element of F then the map given by $f_0(x) = 0$ for all $x \in E$ serves as the zero element of $m(E, F)$. Let $\mathscr{U}$ be the uniformity for F and $\mathscr{B}$ be the translation-invariant base of $\mathscr{U}$. For each $A \in S$ and $U \in \mathscr{B}$ we define

$$M(A, U) = \{(f, g) : (f(x), g(x)) \in U, \forall\, x \in A\}.$$

We claim that the family $\{M(A, U)\}$ $A \in S$ and $U \in \mathscr{B}$ forms a base for a uniformity $\mathscr{U}_S$ for $m(E, F)$ such that each $M(A, U)$ is translation-invariant in the ring $m(E,F)$.

To verify our claim we first prove that $\Delta(m(E, F)) = \{(f, f) : f \in m(E, F)\} \subseteq M(A, U)$ for all $A \in S$ and $U \in \mathscr{B}$. Let $M(B, V)$ be arbitrary in $\{M(A, U)\}$. Let (g, g) be any element of of $\Delta(m(E, F))$, then $(g(x), g(x)) \in \Delta(F)$, $x \in E$,
$\Rightarrow (g(x), g(x)) \in V$, $x \in B$, since $\Delta(F) \subseteq V$ and $B \subseteq E$.
$\Rightarrow (g, g) \in M(B \cdot V.)$

Therefore $\Delta(m(E, F)) \subseteq (M(A, U)$ for all $A \in S$ and $U \in \mathscr{B}$.

Now let $M(A, U) \in \{M(A, U)\}$. Observe that $[M(A, U)]^{-1} = M(A, U^{-1})$. Also, since $U \in \mathscr{B}\ \exists\ V \in \mathscr{B}$ such that $V \subseteq U^{-1}$. It follows easily that

$$M(A, U) \subseteq M(A, U^{-1}) = [M(A, U)]^{-1}.$$

Next let $M(B, V)$ be any member of the family $\{M(A, U)\}$. Then since $V \in \mathscr{B}$ there exists $W \in B$ such that $W^2 \subseteq V$, then it will follow easily that $M(B, W)^2 \subseteq M(B, V)$.

Next let $M(A, U)$ and $M(B, V)$ be any two members of the family $\{M(A, U)\}$ for $A, B \in S$ and $U, V \in \mathscr{B}$. Since $U, V \in \mathscr{B}$ so $\exists\ W \in \mathscr{B}$ such that $W \subseteq U \cap V$ and since $A, B \in S \rightarrow \exists\ C \in S$ such that $A \cup B \subseteq C$. Then it will follow easily that

$M(C, W) \subseteq M(A, U) \cap M(B, V)$. Therefore the family $\{M(A, U)\}$ with $A \in S$ and $U \in \mathscr{B}$ is a base for a uniformity $\mathscr{U}_S$ for $m(E, F)$.

Finally we prove that each $M(A, U)$ in $\{M(A, U)\}$ is translation-invariant under both addition and multiplication in the ring $m(E, F)$. For this, let $(f, g) \in M(A \cdot U \cdot)$ then $(f(x), g(x)) \in U$ for all $x \in A$. Let h be any element of $m(E, F)$. Since U is translation-invariant under both addition and multiplication,

$$(f(x) + h(x), g(x) + h(x)) \in U, \text{ for all } x \in A$$

$$\Rightarrow ((f + h)(x), (g + h)(x)) \in U, \text{ for all } x \in A.$$

$$\Rightarrow (f + h, g + h) \in M(A \cdot U). \qquad \text{Therefore,}$$

$$(h + f, h + g) \in M(A, U).$$

Also

$$(f(x) \cdot h(x), g(x) \cdot h(x)) \in U, \forall\, x \in A$$

$$\Rightarrow ((f \cdot h)(x), (g \cdot h)(x)) \in U, \forall\, x \in A$$

$$\Rightarrow (f \cdot h, g \cdot h) \in M(A, U).$$

Therefore $(h \cdot f, h \cdot g) \in M(A \cdot U \cdot)$.

Thus $M(A, U)$ is translation-invariant under both addition and multiplication. Hence the family $\{M(A, U)\}$ with $A \in S$ and $U \in \mathscr{B}$ is a base for a uniformity $\mathscr{U}_S$ for $m(E, F)$ such that each $M(A, U)$ is translation-invariant in the ring $m(E, F)$. Hence $m_{\mathscr{U}_S}(E, F)$ is uniform ring.

This completes the proof.

Corollary 1—If in Theorem 3, S be the family of all finite subsets of E, then the uniformity $\mathscr{U}_S$ on $m(E, F)$ turns out to be the uniformity of pointwise convergence. This uniformity of pointwise convergence is denoted as $\mathscr{U}_\sigma$.

Corollary 2—If in Theorem 3, $S = \{E\}$ then the uniformity $\mathscr{U}_S$ on $m(E, F)$ turns out to be the uniformity of uniform convergence.

Observation : It appears from the content of Theorems 1 and 3 that the construction of uniform ring $m_{\mathscr{U}_S}(E, F)$ has been obtained under weaker condition imposed on S than the condition imposed on S in the construction of the topological ring $m_{\mathscr{T}_S}(E, F)$.

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