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SAIN IN A NORMED LINEAR SPACE

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In this paper, an attempt has been made to prove theorems on SAIN in a real or complex normed linear space.

1. INTRODUCTION

The concept of simultaneous approximation and interpolation which is norm preserving, was first formulated by Deutsch and Morris (1969). Let M be a dense subspace of a normed linear space X and let $x_1^*, \ldots, x_n^*$ be a finite subset of the dual space X^*. The triple $(X, M, \{x_1^*, \ldots, x_n^*\})$ has property SAIN (Simultaneous Approximation and Interpolation which is Norm preserving), if the following condition is satisfied : For each $x \in X$, and each $\epsilon > 0$, there exists a $y \in M$ such that $\|x-y\| < \epsilon$, $x_i^*(x) = x_i^*(y)$ $(i = 1, \ldots, n)$, and $\|y\| = \|x\|$. In the next section some theorems on SAIN are proved by providing certain conditions, to be satisfied by the entities of the triple $(x, M, \{x_1^*, \ldots, x_n^*\})$.

2. THEOREMS ON SAIN

Some definitions which are necessary in the sequel are given before the first theorem is stated. Let X be a normed linear space, M any dense convex subset of X and Γ any finite dimensional subspace of X^*.

Definition 2.1— The annihilator set L of Γ is given by

$$L = \{x \in X : \gamma(x) = 0, \gamma \in \Gamma\}.$$

Remark : L will always be a subspace of X.

Definition 2.2—$P_L(x)$ denotes the set of all elements of best approximation of $x \in X$, by means of the elements of L, i.e.

$$P_L(x) = \{g_0 \in L : \|x - g_0\| = \inf_{g \in L} \|x - g\|\}.$$

L is said to be semi-Chebyshev if $P_L(x)$ contains atmost one element for every $x \in X$.

Definition 2.3— An element $x \in X$ is said be orthogonal to $y \in X$, written $x \perp y$, if $\|x + \alpha y\| \geqslant \|x\|$, for every real number α. An element $x \in X$ is said to be orthogonal to L, written $x \perp L$ if

$$x \perp y \; (\forall \; y \in L).$$

The condition

$$\|x + \alpha g\| \geqslant \|x\| \; (g \in L, \alpha \text{ real})$$

is equivalent to $\Theta \in P_L(x)$, where Θ is the zero element of X.

Definition 2.4— L^{Θ} is the set of all elements $x \in X$ which are orthogonal to L. In other words,

$$L^{\Theta} = \{x \in X : x \perp L\}.$$

Theorem 2.5— Let X be a real normed linear space, M any dense convex subset of X and Γ any finite dimensional subspace of X^*. (X, M, Γ) has properly SAIN if and only if $L^{\Theta} \subset M$, whenever L is a semi-Chebyshev subspace of X.

PROOF: Holmes and Lambert (1973) have shown that the necessity part is always true, for any subspace L. To prove the sufficiency, let (X, M, Γ) have SAIN and let L be a semi-Chebyshev subspace of X. Let $P_L(x)$ be the set of all best approximations of x in X by means of the elements of L. Let $x \in L^{\Theta}$. It is known that whenever $x \in L^{\Theta}$, that is, whenever $x \perp L$, $\|x - \Theta\| = \inf_{g \in L} \|x - g\|$. Hence $\Theta \in P_L(x)$. Since we have assumed L to be semi-Chebyshev, $P_L(x)$ contains atmost one element. Hence $P_L(x) = \{\Theta\}$. Let $G_x = x - P_L(x)$. Then $G_x = \{x\}$. Holmes and Lambert (1973) have proved in a lemma (Lemma 1) that the triple (X, M, Γ) has property SAIN if and only if whenever $x \in L^{\Theta}$ it follows that $G_x = \overline{M \cap G_x}$. Using this lemma it is obvious that $\overline{M \cap G_x} = \{x\}$. Since G_x is a finite set, $M \cap G_x = \{x\}$. This implies $x \in M$ and therefore $L^{\Theta} \subset M$.

Definition 2.6—Let X be a normed linear space, X^* its dual and $S(X)$ the closed unit ball in X. $x^* \in X^*$ is said to attain its norm at $x \in X$, if $x^*(x) = \|x\|$.

Definition 2.7—A subspace L of a normed linear space X is an *EF* subspace, if $P_L(x)$ for every $x \in X$ is nonempty and finite dimensional.

Definition 2.8—Any subspace M of X is said to be affine if $u, v \in M$ implies that $tu + (1-t)v \in M$ for all real t.

Definition 2.9—A normed linear space is said to be strictly convex if for every, $f, g \in X$ $(f \neq g)$ such that $\|f\| = 1$ and $\|g\| = 1$, $\|h\| < 1$ where $h = 1/2\,(f + g)$.

Remark 2.10 : Holmes and Lambert (1973) have proved that the following conditions are equivalent : When M is a dense affine subspace of a normed linear space X and L is an EF subspace of X

(a) (X, M, Γ) has property SAIN.

(b) each nonzero $\gamma \in \Gamma$ attains its norm solely on M, or not at all,

(c) $L^{\circledR} \subset M$.

(a) and (b) are equivalent in a strictly convex normed linear space. This was shown by Smatkov (1971). (b) and (c) are equivalent, for any L. We have shown that (a) and (c) are equivalent when L is semi-Chebyshev and here M need not be affine. From this, Smatkov's result can be obtained as a corollary.

Corollary 2.11—When X is strictly convex, (X, M, Γ) has SAIN if and only if $L^{\circledR} \subset M$.

Proof : In a strictly convex normed linear space, L is always semi-Chebyshev. The conclusion is now evident from Theorem 2.5.

Remark 2.12 : It is not possible for any arbitrary triple $(X, M, \{x_1^*, \dots, x_n^*\})$ to have SAIN. So a corollary that indicates the existence of SAIN triples is formulated.

Corollary 2.13—Let L be a closed linear subspace of a (real or complex) normed linear space X, of finite co-dimension. $(X, M, L^{\perp})$ has SAIN if and only if $L^{\circledR} \subset M$, whenever L is a semi-Chebyshev subspace of X.

Proof : L is a closed linear subspace of finite co-dimension. Therefore $L = \text{span}\,(f_1, \dots f_n)$, where $f_1, \dots, f_n$ are linearly independent. Consequently

$$L = (L^{\perp})_{\perp} = \{x \in X : f_k(x) = 0, k = 1, 2, \dots, n\}.$$

Since L is semi-Chebyshev, by Theorem 2.5 the result follows.

Remark 2.14 : As we have already stated the above corollary helps us to shown the existence of SAIN triples in normed linear spaces. For example, Singer (1970, p. 314) observes that $C(T)$ which is the space of all real valued continuous functions on an infinite separable compact space T contains semi-Chebyshev subspaces of finite co-dimension. He also observes that $L^1_R\ (T, \nu)$ has Chebyshev subspaces of finite co-dimension, when (T, ν) has n atoms. In all these spaces the existence of semi-Chebyshev and Chebyshev subspaces implies the existence of SAIN triples. As Singer (1970, p. 114) observes, every normed linear space has at least one dense subspace, say M. So when we look for SAIN triples we can assume the existence of a dense subspace. This dense subspace M is semi-Chebyshev, since $P_M(x) = \phi\ (x \in X \setminus M)$ and $P_M(x) = \{x\}\ (x \in M)$. This fact is exploited in the next theorem.

Theorem 2.15— Let M be a dense subspace of a (real or complex) normed linear space X and Γ any finite subset of X^*. (X, M, Γ) has SAIN if whenever $\Theta \in P_L(x)$, $P_M(x) \neq \phi$.

The condition that whenever $\Theta \in P_L(x)$, $P_M(x) \neq \phi$, $\forall\ x \in X$, is not sufficient for (X, M, Γ) to have property SAIN, as the following example shows. Let $X = C[0, 1]$ and let M denote the space of μ-simple functions on $[0, 1]$. Then M is a dense subalgebra of X. Let $x^* \in C[0, 1]^*$, the dual of $C[0, 1]$, such that $x^*(x) = X(1)$, for every $x \in X$. Now x^* is an evaluation functional at 1. Deutsch and Morris (1969) have shown that if $C(T)$ is the space under consideration, M any dense subalgebra of $C(T)$ and δ_{t_i} are point evaluation functionals at t_i $(i = 1, 2, \ldots, n)$ then $(C(T), M, \{\delta_{t_i}\}_{i=1}^{n})$ has property SAIN. In the present context it follows that $(X, M, \{x^*\})$ has property SAIN. Let $f \in C[0, 1]$ such that $f(t) = t^2$, $t \in [0, 1]$. Then

$$L = \{g \in X : x^*(g) = 0\}$$

and

$$x^*(f - \Theta) = x^*(f) = f(1) = \|f\|$$

$$x^*(g) = 0 \text{ for all } g \in L,$$

$$\|x^*\| = 1.$$

Thus, by a Theorem of Singer [page 18], $\Theta \in P_L(f)$. But f is not a μ-simple function and $f \notin M$. Hence $P_M(f)$ is empty!

Proof of Theorem 2.15: Let $x \in L^{\Theta}$. Then $\Theta \in P_L(x)$. Whenever $\Theta \in P_L(x)$, we have $P_M(x) \neq \phi$. That is, $P_M(x) = \{x\}$, since M is a dense subspace of X. Therefore $L^{\Theta} \subset M$, which implies that (X, M, Γ) has SAIN.

Definition 2.16—Singer (1970, p. 88) gives the following definition. Let X be a (real or complex) normed linear space and let K be a convex set containing the origin as an internal point. Then the real valued function τ, defined for $x, y \in X$ by

$$\tau(x, y) = \lim_{t \to 0^+} \frac{\|x + ty\| - \|x\|}{t}$$

is called the tangent function of K.

Theorem 2.17—Let M be a dense subspace of a normed linear space X (real or complex) and Γ a finite subset of X^*. (X, M, Γ) has property SAIN, whenever $\tau(x, g) < 0$, $g \in L$, for every $x \in X \setminus M$.

The condition that $\tau(x, g) < 0$, for all $g \in L$ and for every $x \in X \setminus M$ is not sufficient for the triple (X, M, Γ) to have SAIN. For, if $X = C[0, 1]$ and M is the

set of all polynomials in $C[0, 1]$, then M is a dense subalgebra of X. Let $x^* \in X^*$, such that $x^*(x) = x(1)$ for all $x \in X$. x^* is a point evaluation functional. Deutsch and Morris (1969) have shown that if $C(T)$ is the space under consideration, M any dense subalgebra of $C(T)$ and δ_{t_i} are point evaluation functionals at t_i $(i = 1, 2, \ldots, n)$ then $(C(T), M, \{\delta_{t_i}\}_{i=1}^{n})$ has property SAIN. In the present context, it follows that $(C[0, 1]. M, \{x^*\})$ has property SAIN. Let $f \in C[0, 1]$ be defined by $f(t) = 1$ in the closed interval $[1/2, 1]$ and $0 \leqslant f(t) \leqslant 1$ otherwise.
Then $f \in X \setminus M$ and

$$x^*(f - \Theta) = x^*(f) = f(1) = \|f\|.$$

Further

$$L = \{x \in X : x^*(x) = 0\}.$$

Therefore

$$x^*(x) = 0 \text{ for all } x \in L$$

and

$$\|x^*\| = 1.$$

Singer (1970, p. 18) has proved that if there exists $x^* \in X$ such that $x^*(x) = 0$ for all $x \in L$, $\|x^*\| = 1$ and $x^*(f - \Theta) = \|f\|$, then $\Theta \in P_L(f)$. Further, he has established that $\Theta \in P_L(f)$ and $\tau(f, g) \geqslant 0$ are equivalent conditions [p. 88]. Hence $\tau(f, g) < 0$, for all $g \in L$ is not true.

Proof of Theorem 2.17 : Let $x \in X \setminus M$. As t decreases, $\|x + tg\| - \|x\|/t$ also decreases for the positive variable t. (See reference 4, p. 351, Proposition 1). Since $\tau(x\, g) < 0$, for every $x \in X \setminus M$ and for the particular case when $t = 1$, we get $\|x + g\| < \|x\|$. This shows that x is not orthogonal to L. That is, every x in $X \setminus M$ is not in L^{Θ}. This L^{Θ} if it exists, is in M. The condition $L^{\Theta} \subset M$ being satisfied, (X, M, Γ) has property SAIN.

Definition 2.18—Singer (1970, p. 135) gives the following definitions.

Let X be a (real or complex) normed linear space, G a linear subspace of X, $\phi_1, \ldots, \phi_n \in G^*$ and $x \in X$. The interpolatory element of best approximation of x is any element g_0 in G with the properties

$$\phi_i(g_0) = C_i \ (i = 1, \ldots, n), \ C_i\text{'s are scalars}$$

and

$$\|x - g_0\| = \inf_{g \in G} \|x - g\|.$$

$$\phi_i(g) = C_i \ (i = 1, \ldots, m).$$

This definition is used in the next theorem.

Notation : Let $W_x = \{x\} \oplus L$, where $x \in M$. Let $\phi \in X^*$ and define $B_x(y)$ to be the set of all interpolatory elements of best approximation of y by means of elements of W_x, taking ϕ to be the scalar value. $B_x(y)$ consists of elements g_0 in W_x such that

$$\phi(g) = \|\phi\|$$

and

$$\|y - g\| = \inf_{g \in W_x} \|y-g\|.$$

$$\phi(g) = \|\phi\|$$

Theorem 2.19—Let ϕ, W_x and $B_x(y)$ be as defined and let M be any dense subspace of X. (X, M, ϕ) has SAIN if for each $x \in X \setminus M$, $B_x(x)$ is empty.

In the example given after Theorem 2. 17, x^* attains its norm at $f \in B_f(f)$. But the triple $(X, M, \{x^*\})$ has property SAIN.

PROOF : Whenever ϕ attains its norm at $x \in X \setminus M$, $\phi(x) = \|\phi\|$. Since $x \in W_x$, it belongs to $B_x(x)$ if $\phi(x) = \|\phi\|$. The other condition is true for every element y in W_x with $\phi(y) = \|\phi\|$. $B_x(x)$ is empty when x is nonextremal, that is, when $\phi(x) \neq \|\phi\|$. Deutsch and Morris (1969) has shown that (X, M, ϕ) has SAIN when ϕ does not attain its norm at $x \in X \setminus M$. Hence the theorem is proved.

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