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THE PRIME RADICAL IN NEAR-RINGS

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In this paper the prime radical of an ideal in a zero-symmetric near-ring is characterized in terms of "m^*-systems" and 'm^*-sequences'.

§1. Throughout by a near-ring we mean a zerosymmetric right near-ring. If $A, B \subseteq N$, then $A : B = \{n \in N | nB \subseteq A\}$ and $AB = \{ab | a \in A, b \in B\}$. If $A = \{a\}$, we write aB for AB. For any positive integer k, in the usual way we define A^k. For any $x \in N$, $< x >$ denotes the ideal generated by x. Throughout N stands for a near-ring and for the preliminaries we shall refer to Pilz (1977).

We recall an ideal K of N is prime if $IJ \subseteq K$, I and J are ideals of N, implies either $I \subseteq K$ or $J \subseteq K$. We remark that if P is an ideal of N, then (i) P is prime, (ii) $a < b > \subseteq P$, where $a, b \in N$, implies $a \in P$, or $b \in P$, and (iii) $A_1A_2 \ldots A_n \subseteq P$, A_i's are ideals of N for $1 \leqslant i \leqslant n$, implies $A_i \subseteq P$ for some i are equivalent (Chaudhary 1973). An ideal S of N is semiprime if for any ideal J of N, $J^2 \subseteq S$ implies $J \subseteq S$. We remark that if S is an ideal of N, then (i) S is semiprime, (ii) $a < a > \subseteq S$, $a \in N$, implies $a \in S$, and (iii) $A^k \subseteq S$, A is an ideal of N and k is a positive integer, implies $A \subseteq S$ are equivalent (Maxson 1967). If S is a semiprime ideal of N, then S contains all nilpotent ideals of N.

§2. We start with the following definition.

Definition—A subset M of N is said to be an m^*-system if for any $a, b \in M$, there exists an element $b_1 \in < b >$ such that $ab_1 \in M$.

This definition generalizes the definition of an m-system, which was introduced in Van der Walt (1964). We remark that (i) if P is an ideal of N, then P is prime if and only if $N \setminus P$ is an m^*-system. (ii) If M is a non-empty m^*-system and if I is an ideal of N such that $M \cap I = \phi$, then there exists a prime ideal P containing I such that $P \cap M = \phi$.

Definition—A subset S of N is said to be an sp^*-system if for any $s \in S$, there exists an element $s_1 \in < s >$ such that $ss_1 \in S$.

This defiinition implies the definition of an sp-system in Maxson (1967). We remark that if S is an ideal of N, then S is semiprime if and only if $N \setminus S$ is an sp^*-system.

Definition—A sequence $a_0, a_1, \ldots, a_n, \ldots$ of elements in N is called an m^*-sequence if $a_n \in a_{n-1} < a_{n-1} >$ for all $n \geqslant 1$.

Clearly the m-sequence defined in Ramakotaiah (1967) is an m^*-sequence. But in the ring of even integers, the sequence $\{2^{2^n}\}_{n \geqslant 0}$ shows that every m^*-sequence need not be an m-sequence (Ramakotaiah 1967).

We remark that if $\{a_n\}_{n \geqslant 0}$ is an m^*-sequence in N, then the set $\{a_n / n \geqslant 0\}$ is an m^*-system. Parallel to the Prop. 2.92 in Pilz (1977), we have for any sp^*-system S and $a \in S$, there exists an m^*-sequence $\{a_n\}_{n \geqslant 0}$ in S starting with a (i.e. $a = a_0$). If S is an sp^*-system and if $a \in S$, then there exists an m^*-system M such that $a \in M \subseteq S$.

We recall that, for a proper ideal I of N, the prime radical of I is the intersection of all prime ideals containing I and it is denoted by $\mathscr{P}(I)$. The prime radical of N is defined as $\mathscr{P}(\{0\})$.

We now prove our main results.

Theorem 1—If I is a proper ideal of N, then

$$\mathscr{P}(I) = \left\{ a \in N / \begin{array}{c} \text{every } m^*\text{-sequence } \{a_n\}_{n \geqslant 0} \text{ starting} \\ \text{with } a \text{ meets } I \end{array} \right\}.$$

PROOF: Let X be the right-hand side set. Let $a \in N$ with $a \notin \mathscr{P}(I)$. Then there exists a prime ideal P containing I such that $a_0 = a \notin P$. Since P is prime, $N \setminus P$ is an m^*-system and so $N \setminus P$ is an sp^*-system. Now there exists an m^*-sequence in $N \setminus P$ starting with a_0 which does not meet P and so, evidently it does not meet I.

Thus $a \notin X$. Hence $X \subseteq \mathscr{P}(I)$.

Conversely suppose $a \in N$ with $a \notin X$. Then there is an m^*-sequence $\{a_n\}_{n \geqslant 0}$ such that $a_0 = a$ and $a_n \notin I$ for all n. Write $M = \{a_n / n \geqslant 0\}$. Since $M \cap I$ is empty and M is an m^*-system, there is a prime ideal P such that $P \supseteq I$ and $P \cap M$ is empty. Clearly $a \notin P$ and so $a \notin \mathscr{P}(I)$. Thus $X \supseteq \mathscr{P}(I)$. Hence the theorem.

The proof of the following lemma is straight forward.

Lemma 1— If $\{a_n\}_{n \geqslant 0}$ is an m^*-sequence in a commutative near-ring N, then each a_n is a finite sum of elements of the form $y + xa_0^m - y$, where $y \in N$ and either $x \in N$ or x is an integer.

Theorem 2—If I is an ideal in a commutative near-ring N, then $\mathscr{P}(I) = \{a \in N / a^n \in I \text{ for some positive integer } n\}$.

PROOF : Let $a \in N$ with $a^m \in I$ for some positive integer m. Let $\{a_n\}_{n \geqslant 0}$ be an m^*-sequence starting with a. Now by Lemma 1, each a_n is a finite sum of elements of the form $y + xa^{2^n} - y$, where $y \in N$ and $x \in N$ or x is an integer. If k is a positive integer such that $2^k \geqslant m$, then $a^{2^k} \in I$ and hence $y + xa^{2^k} - y \in I$. Consequently $a_k \in I$. Thus every m^*-sequence $\{a_n\}_{n \geqslant 0}$ with $a_0 = a$ meets I and hence $a \in \mathscr{P}(I)$ by Theorem 1. Therefore $\mathscr{P}(I)$ contains the right hand side set. The converse is easy and will be omitted.

Definition—An element a in a near-ring N is said to be strongly nilpotent if every m^*-sequence $\{a_n\}_{n \geqslant 0}$ with $a_0 = a$ vanishes (i.e., there is a positive integer n such that $a_k = 0$ for all $k \geqslant n$).

Clearly every strongly nilpotent element in N is nilpotent. But the following example shows that the converse is not true.

Example : Consider the ring of all 3×3 matrices over the ring of integers. Write

$$a_0 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and } x = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Define $a_1 = a_0 xa_0, \ldots, a_n = a_{n-1} xa_{n-1}, \ldots$. Now $a_n \neq 0$ for all $n \geqslant 0$ and hence a_0 is not strongly nilpotent. But $a_0^3 = 0$. This shows that a_0 is nilpotent.

The following is a consequence of Lemma 1.

Lemma 2— If a is an element in a commutative near-ring N, then a is strongly nilpotent if and only if it is nilpotent.

Theorem 3— The prime radical of N is the set of all strongly nilpotent elements in N.

Proof follows from Theorem 1.

Theorem 4— Let I be an ideal of N. Then

$$\mathscr{P}(I) = \left\{ a \in N \middle| \begin{array}{c} \text{every } m^*\text{-system that contains } a \text{ must} \\ \text{contains an element of } I \end{array} \right\}.$$

PROOF : Let X be the right-hand side set. Let $a \in \mathscr{P}(I)$. If $a \notin X$, then there exists an m^*-system M with $a \in M$ and $M \cap I = \phi$. This implies there exists a prime ideal P such that $P \supseteq I$ and $M \cap P = \phi$. So $a \notin P$. Therefore $a \notin \mathscr{P}(I)$, a contradiction.

Conversely suppose $a \in X$. If $a \notin \mathscr{P}(I)$, then there exists a prime ideal P containing I such that $a \notin P$. This implies $a \in N \setminus P$ and $N \setminus P$is an m^*-system. Since $a \in X$, we have $(N \setminus P) \cap I \neq \phi$, a contradiction. Hence the theorem.

Corollary 1— If N is a near-ring, then

$$\mathscr{P}(\{0\}) = \{a \in \dot{N} / \text{ every } m^*\text{-system that contains a contains } 0\}.$$

Corollary 2— (Pilz 1977, p. 162)— $\mathscr{P}(\{0\})$ is a nil ideal of N.

Proof follows from Theorem 3.

Corollary 3—The prime radical of a commutative near-ring N is the set of all nilpotent elements in N.

Proof follows from Corollary 2, Lemma 2 and Theorem 3.

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