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SOME ABELIAN THEOREMS FOR THE DISTRIBUTIONAL *H*-TRANSFORMATION

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Some Abelian theorems are given for the conventional generalization of Whittaker transformation by Moharir and Saxena (1979). This transformation has been extended to a certain class of generalized functions (distributions) by the authors recently. In the present paper we establish some Abelian theorems for this distributional *H*-transformation.

1. INTRODUCTION

Some well-known generalizations of the conventional Laplace transformation have been extended to certain classes of generalized functions (distributions) by various mathematicians from time to time. The *H*-transformation (generalized Whittaker transformation) due to Moharir and Saxena (1977) defined by the convergent integral

$$F(s) = \int_0^\infty e^{\lambda st} H_{p,q}^{n,0}\left[\nu (st)^\mu \,\middle|\, \begin{matrix}(a_p, \alpha_p)\\(b_q, \beta_q)\end{matrix}\right] f(t)\, dt \qquad \ldots(1.1)$$

in which Re $(s), \lambda, \nu > 0$, $\mu \equiv \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j > 0$ and $H_{p,q}^{m,n}[z]$ is a Fox's *H*-function [Srivastava *et al.* (1982)], has recently been extended to a certain class of generalized functions by Malgonde and Saxena (1981, 1982). Similar work in the distributional sense is also due to Sinha (1981) and Tiwari and Ko (1982), who studied the generalized Whittaker transformation defined by

$$F(s) = \int_0^\infty (st)^{\sigma-(1/2)} e^{-1/2\rho st} W_{k,m}(\rho\, st) f(t)\, dt \qquad \ldots(1.2)$$

due to Srivastava (1968) and studied further by Srivastava and Vyas (1969). A similar type of work (in the distributional sense) due to Joshi and Saxena (1983) is also an interesting unification of the various known generalizations of the Laplace transformation. The object of the present work is to establish some Abelian theorems for the distributional H-transformation. (By an "initial (final—) value theorem" we mean a theorem that relates the initial (final—) value of a distribution to the final (initial—) value of the transformation).

Our notation is the same as that used in Malgonde and Saxena (1981). Throughout this work I denotes the open interval $(0, \infty)$, and t, σ are real variables restricted to I; $s = \sigma + i\omega$ is a complex variable. D_I denotes the space of smooth functions whose supports are compact subset of I. We assign to D_I the topology that makes its dual D_I' the space of Schwartz distributions on I. $\mathscr{E}_I$ and $\mathscr{E}_I'$ are respectively the space of smooth functions on I and the space of distributions having compact supports with respect to I. Let a be a suitably fixed real number, b be a real (positive) number. Let $H_{a,b}$ be the space of all complex valued smooth functions $\phi(t)$ on I for which

$$\sup_{0<t<\infty} \left| e^{-bt} t^{a} \left(t \frac{d}{dt} \right)^{r} \phi(t) \right| < \infty$$

for each nonnegative integer r, and the dual space $H'_{a,b}$ of $H_{a,b}$ consists of all linear continuous functionals on $H_{a,b}$ as in Malgonde and Saxena (1981).

It is easy to check that the space $D_I \subset H_{a,b}$ and that the topology of D_I is stronger than that induced on it by $H_{a,b}$. Hence the restriction of any $f \in H'_{a,b}$ to D_I is in D_I'. It is also easy to check that $\mathscr{E}_I' \subset H'_{a,b}$. If f is a generalized function, then $f(t)$ is used to indicate that the testing functions on which f is defined have t as the variable. If an ordinary function has continuous derivatives of all orders at every point of its domain, we shall call it a smooth function. In this work we shall assign limits to certain regular distributions as given by Zemanian (1966).

2. An Initial-Value Theorem for the Distributional H-Transformation

Moharir and Saxena (1979) have proved Abelian theorems for the conventional H-transformation.

We now consider the distributional H-transformation. Let a be a suitably fixed real number, b be a real (+ ve) number and $\lambda > 0$. If $f \in H'_{a,b}$, then the distributional H-transformation exists for at least $0 < \operatorname{Re} s < b/\lambda$, $\operatorname{Re}(a + \mu c) \geq 0$ and is given by

$$F(s) = \left\langle f(t), e^{\lambda st} H_{p,q}^{m,0} \left[\nu (st)^{\mu} \middle| \begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \right] \right\rangle. \qquad \ldots(2.1)$$

Before going to extend the result of initial value theorem of Moharir and Saxena (1979) to the distributions, we shall first extend somewhat the definition of the spaces $H_{a,b}$ and $H'_{a,b}$ given by Malgonde and Saxena (1981). Previously, these definitions stipulated that b be a real (+ ve) number. Here we shall merely require that b be a extended real (+ ve) number. This allows us to consider the spaces $H_{a,\infty}$ and $H'_{a,\infty}$. The analysis and results of Malgonde and Saxena (1981) extend directly to these spaces. For example, if $f \in H'_{a,\infty}$ and a be a suitably fixed real number, and the distributional H-transformation $F(s)$ of f, which is defined by (2.1), exists and is analytic for $0 < \text{Re}\, s < \infty$. Also for any $B > 0$ the restriction of $f \in H'_{a,\infty}$ to $H_{a,B}$ is in $H'_{a,B}$. The members of $H'_{a,\infty}$ do possess the following property which in general the members of $H'_{a,B}$ can not have.

Lemma 1—Let a be a suitably fixed real number. If $f \in H'_{a,\infty}$ and $F(s)$ is given by (2.1), then

$$\lim_{\sigma \to \infty} \sigma^{\gamma+1} F(\sigma) (-\lambda)^{\gamma+1} = 0 \text{ wherever } \gamma > -1.$$

PROOF: Using the boundedness property of the generalized functions from [Malgonde and Saxena (1981), § 3, Property (iv)], one can easily prove that

$$| F(\sigma) | = 0 \text{ for } 0 < \sigma < \infty.$$

This shows that $F(\sigma) = 0$ for $0 < \sigma < \infty$, which immediately implies the desired conclusion.

From the initial value Theorem of Moharir and Saxena (1979), Lemma 1 and the fact that the distributional H-transformation (2.1) contains the transformation (1.1) as a special case [Malgonde and Saxena (1981), § 3, property (v)] directly imply the following principal conclusion for this section.

Theorem 1— Let a be a suitably fixed real number and $\gamma > -1$. Assume that $f = f_1 + f_2$, where f_1 is an ordinary function satisfying the hypothesis of initial value Theorem of Moharir and Saxena (1979), and where $f_2 \in H'_{a,\infty}$. If $F(s)$ is defined by (2.1), then

$$\lim_{\sigma \to \infty} \sigma^{\gamma+1} F(\sigma) (-\lambda)^{\gamma+1} = H[\nu/(-\lambda)^{\mu}] \lim_{t \to 0+} t^{-\gamma} f(t)$$

where

$$H[\nu/(-\lambda)^{\mu}] = H^{m,1}_{p+1,q} \left[\nu \mid (-\lambda)^{\mu} \mid \begin{matrix} (-\gamma, \mu)\,(a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \right]. \qquad \ldots(2.2)$$

PROOF: Note that, if the support of f_2 is contained in $[y, \infty]$ for some $y > 0$ then f_2 must be in $H'_{a,\infty}$ if it is in some $H'_{a,B}$ $(B > 0)$; moreover, by the convention from Zemanian (1966),

$$\lim_{t \to 0+} t^{-\gamma} f_2(t) = 0$$

so that

$$\lim_{t \to 0+} t^{-\gamma} f(t) = \lim_{t \to 0+} t^{-\gamma} f_1(t) = A.$$

Moreover, from [Malgonde and Saxena (1981), § 3, Property (v)], the distributional H-transformation F_1 of f_1 equals the conventional H-transformation of f_1. Therefore, by initial value Theorem of Moharir and Saxena (1979),

$$\lim_{\sigma \to \infty} \sigma^{\gamma+1} F_1(\sigma) (-\lambda)^{\gamma+1} = A . H [\nu/(-\lambda)^{\mu}].$$

Denoting the distributional H-transformation of f_2 by F_2. We have from Lemma 1, that

$$\lim_{\sigma \to \infty} \sigma^{\gamma+1} F_2(\sigma) (-\lambda)^{\gamma+1} = 0.$$

Since $F(\sigma) = F_1(\sigma) + F_2(\sigma)$

$$\lim_{\sigma \to \infty} \sigma^{\gamma+1} F(\sigma) (-\lambda)^{\gamma+1} = H [\nu/(-\lambda)^{\mu}]. A$$

This completes the proof.

3. A Final-Value Theorem for the Distributional H-Transformation

In this section we shall extend the result of final-value Theorem from Moharir and Saxena (1979) to the distribution and obtain the final-value theorem for the distributional H-transformation. It relates the behaviour of $f(t)$ as $t \to \infty$ to the behaviour of $F(\sigma)$ as $\sigma \to 0+$. It is the final behaviour of $f(t)$ which is being considered. The following lemma is used in extending the result of final-value Theorem from Moharir and Saxena (1979).

Lemma 2— Let a be a suitably fixed real number, b be a real (+ ve) number and $\lambda > 0$. If $f \in H'_{a,b}$ and $F(s)$ is defined by (2.1), then

$$\lim_{\sigma \to 0+} \sigma^{\gamma+1} F(\sigma) (-\lambda)^{\gamma+1} = 0 \text{ whenever } \gamma > -1.$$

Proof: It is clear that for Re $(a + \mu c) \geqslant 0$ and $0 <$ Re $s < b/\lambda$, $|F(\sigma)|$ is bounded. This shows that $F(\sigma)$ is bounded which immediately implies the desired conclusion. We are now ready to state and prove the final value theorem for the distributional H-transformation.

Theorem 2— Let a be a suitably fixed real number, b be a real (+ ve) number, $\lambda > 0$ and $\gamma > -1$. Assume that $f = f_1 + f_2$ where f_1 is an ordinary function satisfying the hypothesis of final-value Theorem from Moharir and Saxena (1979), and f_2 is a distribution satisfying the hypothesis of Lemma 2. Let $F(\sigma)$ be defined by (2.1), and $H[\nu/(-\lambda)^{\mu}]$ defined by (2.2). Then,

$$\lim_{\sigma\to 0+} \sigma^{\gamma+1} F(\sigma)(-\lambda)^{\gamma+1} = A.\ H[\nu/(-\lambda)^{\mu}].$$

Proof: Suppose $f(t)$ is decomposed into $f(t) = f_1(t) + f_2(t)$. Note that, if $f_2(t)$ has its support contained in $[0, T]$ for some $T < \infty$, then $f_2(t)$ is in $H'_{a,b}$. Moreover, by the convention from Zemanian (1966) we have

$$\lim_{t\to\infty} t^{-\gamma} f_2(t) = 0.$$

In this case we can set

$$\lim_{t\to\infty} t^{-\gamma} f(t) = \lim_{t\to\infty} t^{-\gamma} f_1(t) = A. \qquad \ldots(3.1)$$

Since $F(\sigma) = F_1(\sigma) + F_2(\sigma)$

where $F_1(\sigma)$ and $F_2(\sigma)$ are defined in (1.1) and (2.1) respectively. Therefore

$$\begin{aligned}
\lim_{\sigma\to 0+} \sigma^{\gamma+1} F(\sigma)(-\lambda)^{\gamma+1} &= \lim_{\sigma\to 0+} \sigma^{\gamma+1} F_1(\sigma)(-\lambda)^{\gamma+1} \\
&\quad + \lim_{\sigma\to 0+} \sigma^{\gamma+1} F_2(\sigma)(-\lambda)^{\gamma+1} \\
&= \lim_{\sigma\to 0+} \sigma^{\gamma+1} F_1(\sigma)(-\lambda)^{\gamma+1} \text{ (by Lemma 2)} \\
&= H[\nu/(-\lambda)^{\mu}]\, t^{-\gamma} f_1(t) \\
&= H[\nu/(-\lambda)^{\mu}] \lim_{t\to\infty} t^{-\gamma} f(t) \text{ by (3.1)} \\
&= H[\nu/(-\lambda)^{\mu}].\ A \text{ (by (3.1)}.
\end{aligned}$$

Hence

$$\lim_{\sigma\to 0+} \sigma^{\gamma+1} F(\sigma)(-\lambda)^{\gamma+1} = A.\ H[\nu/(-\lambda)^{\mu}].$$

This completes the proof.

On specializing the parameters suitably in the kernel of the H-transformation defined by (1.1), we obtain as particular case the corresponding results for Srivastava's generalized Whittaker transformation defined by (1.2).

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