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REFLECTION AND REFRACTION OF PLANE HARMONIC SHEAR WAVES AT A SOLID-SANDY BILATERAL INTERFACE

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This paper is concerned with the reflection and refraction phenomena of elastic waves due to incident shear wave at a plane interface between solid-sandy half-spaces in welded contact and under initial stress of different types. The numerical values of the amplitude ratios of both the reflected and refracted waves have been calculated for different values of the angles of incidence, the initial stress and the sandiness. The nature of the variations of these amplitudes is shown in the graphs for different parametric values and manv interesting results characterizing the initial stress and the sandiness in addition to the angles of incidence are distinctly marked.

1. INTRODUCTION

The study of the reflection and refraction phenomena of elastic waves at an interface between two elastic half-spaces having different types of material properties and under different types of initial stresses is of considerable interest in the field of seismology, in particular seismic prospecting, as the earth's surface might be supposed to be consisting of initially stressed layers having material properties of different types.

The usual method of potential breaking is not applicable here to explain correctly the reflection and refraction phenomena of elastic waves as the initial stress acting in the medium makes it anisotropic in nature. Taking into account the anisotropic nature of the medium, a generalised approach followed by Achenbach (1973) is adopted here to explain the reflection and refraction phenomena of elastic waves.

The problems of the reflection and refraction of elastic waves in layered media, free of initial stress have been discussed by several authors including Knoot (1899), Gutenberg (1944), Jefferys (1944), Henneke (1962), Thapliyal (1972) and others.

Dey and Addy (1979) have studied the problem of reflection and refraction of elastic waves under initial stress. Chattopadhyay *et al.* (1982) have obtained the reflection coefficients of elastic waves in a sandy half-space. But their solutions do not work on account of the anisotropic nature of the medium due to the presence of initial stress

Here we have investigated the nature of the variations of the amplitudes of both the reflected and refracted waves due to incident plane shear wave at a bilateral interface between two solid-sandy initially stressed half-spaces. The numerical values of the amplitude ratios of both the reflected and refracted waves are presented in the graphs, and they show these amplitude ratios vary with respect to the initial stress and the sandiness in addition to the angles of incidence.

2. Formulation of the Problem

We have considered the upper elastic half-space (M_2) to be sandy and the lower half-space (M_1) free of sandiness and both the media under different types of initial stress be in contact at the bilateral interface $x_2 = 0$. The co-ordinate axes x_1 and x_2 are placed in relation to the two elastic half-spaces as shown in Fig. 1.

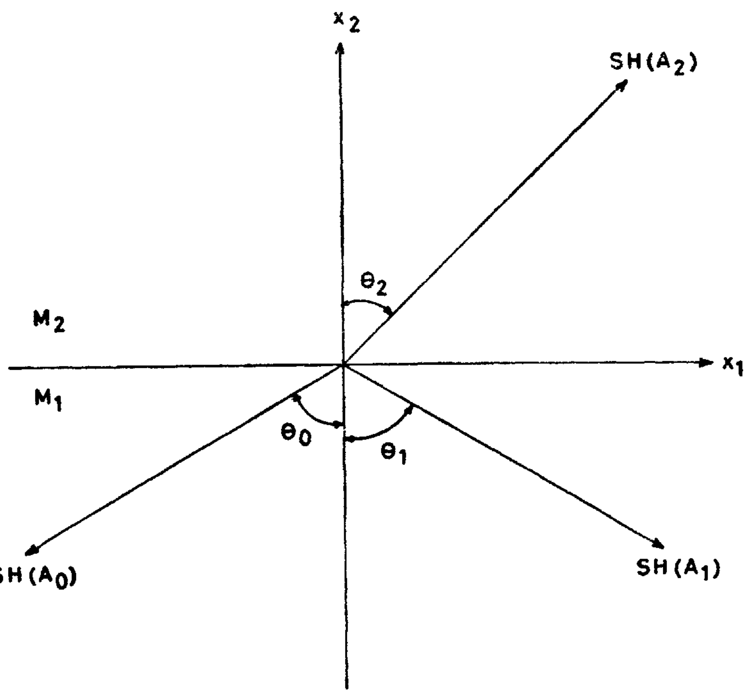

Fig. 1. Reflection and refraction of SH waves at a solid-sandy bilateral interface.

Let S_{11} and S_{22} be the initial stress components acting in the x_1 and x_2 direction respectively and η (> 1) the sandiness of the medium.

Since our problem is concerned about the propagation of SH waves, the existing dynamical equation of motion under initial stress is given by Biot (1965) as

$$s_{31,1} + s_{23,2} - S_{11}\,\omega_{13,1} + S_{22}\,\omega_{32,2} = \rho\,\ddot{u}_3 \qquad \ldots(2.1)$$

where s_{ij} and ω_{ij} are the incremental stress and rotational components respectively. The operators, i and $\cdot\cdot$ represent the differentiation once with respect space-co-ordinate and twice with respect time co-ordinate respectively.

The incremental stress components s_{ij} in terms of strain components e_{ij} in an initially stressed sandy medium are given by

$$\left.\begin{aligned} s_{23} &= 2\eta Q_1\, e_{12} \\ s_{31} &= 2\eta Q_2\, e_{31} \end{aligned}\right\} \qquad \ldots(2.2)$$

where the elastic moduli Q_1 and Q_2 are functions of the initial stresses. These relations coincide with [Biot 1965, Ch. 2, eqn. (6.1)] when $\eta = 1$ i.e when the medium is free of sandiness.

As an SH wave incident at the interface will give rise to both the reflection and refraction of SH waves, the dynamical equations of motion is terms of displacement components are

$$\left(\eta Q_1 + \frac{S_{22}}{2}\right) u^{(n)}_{3,22} + \left(\eta Q_2 + \frac{S_{11}}{2}\right) u^{(n)}_{3,11} = \rho\, \ddot{u}^{(n)}_3 \qquad \ldots(2.3)$$

where the displacement vector $\mathbf{U}^{(n)} = (0, 0, u_3^{(n)})$ is given by

$$\mathbf{U}^{(n)} = A_n\, \mathbf{d}^{(n)}\, e^{ik_n (\mathbf{x} \cdot \mathbf{p}^{(n)} - c_n t)}. \qquad \ldots(2.4)$$

Here different values of n are associated with the incident, reflected and refracted SH waves, Also $\mathbf{d}^{(n)}$ and $\mathbf{p}^{(n)}$ are the unit displacement and the unit propagation vectors respectively with k_n denoting the corresponding wave number.

For the upper medium (M_2), we shall be using primed quantities to represent the physical constants and the field quantities.

The transverse wave velocities in the lower medium (M_1) referred to the index n are obtained by substituting (2.4) into (2.3) and writing $\eta = 1$ and are given by

$$c_n^2 = c_T^2 = \frac{R_1 + (R_2 - R_1)\,\{p_1^{(n)}\}^2}{\rho} \qquad \ldots(2.5)$$

where

$$R_1 = Q_1 + \frac{S_{22}}{2},\; R_2 = Q_2 + \frac{S_{11}}{2}\,. \qquad \ldots(2.6)$$

Similarly, the transverse wave velocity in upper medium (M_2) is given by

$$c_n^2 = c_T'^2 = \frac{R_1' + (R_2' - R_1')\,\{p_1^{(n)}\}^2}{\rho'} \qquad \ldots(2.7)$$

where

$$R_1' = \eta' Q_1' + \frac{S_{22}'}{2}, \quad R_2' = \eta' Q_2' + S_{11}'/2. \qquad \ldots(2.8)$$

The existing boundary forces per unit initial area in the lower medium referred to the index n are

$$\Delta f_3^{(n)} = R_1 u_{3'2}^{(n)}$$

which may be written as

$$\Delta f_3^{(n)} = i\, R_1\, A_n\, k_n\, d_3^{(n)}\, p_2^{(n)}\, e^{i\lambda_n} \qquad \ldots(2.9)$$

where

$$\lambda_n = k_n\,(x_1\, p_1^{(n)} + x_2\, p_2^{(n)} + x_3\, p_3^{(n)} - c_n t). \qquad \ldots(2.10)$$

Similar boundary forces in the upper medium are

$$\Delta f_3'^{(n)} = i\, R_1'\, k_n\, A_n\, d_3^{(n)}\, p_2^{(n)}\, e^{i\lambda_n}. \qquad \ldots(2.11)$$

3. Solution of the Problem

Let an SH wave making an angle θ_0 with the normal to the interface be incident at the plane $x_2 = 0$ and hence it gives rise to both the reflection and refraction of SH waves (Fig. 1).

Thus,

incident wave : $n = 0,\ c_0 = c_T,\ \mathbf{d}^{(0)} = (0, 0, 1),$

$$\mathbf{p}^{(0)} = (\sin\theta_0, \cos\theta_0, 0).$$

reflected wave : $n = 1,\ c_1 = c_T,\ \mathbf{d}^{(1)} = (0, 0, 1),$

$$\mathbf{p}^{(1)} = (\sin\theta_1, \cos\theta_1, 0).$$

refracted wave : $n = 2,\ c_2 = c_T',\ \mathbf{d}^{(2)} = (0, 0, 1),$

$$\mathbf{p}^{(2)} = (\sin\theta_2, \cos\theta_2, 0).$$

The displacement component u_3 and the boundary forces Δf_3 in the lower medium are

$$\left.\begin{aligned} u_3 &= u_3^{(0)} + u_3^{(1)} \\ \Delta f_3 &= \Delta f_3^{(0)} + \Delta f_3^{(1)} \end{aligned}\right\} \qquad \ldots(3.1)$$

and those in the upper medium are

$$\left.\begin{aligned} u_3' &= u_3'^{(2)} \\ \Delta f_3' &= \Delta f_3'^{(2)} . \end{aligned}\right\} \qquad \ldots(3.2)$$

Boundary conditions—As these two media are in welded contact, the displacements and the boundary forces are continuous at the interface $x_2 = 0$.

Therefore,

$$\left.\begin{aligned} u_3 &= u_3' \\ \Delta f_3 &= \Delta f_3' \end{aligned}\right\} \text{ at } x_2 = 0. \qquad \ldots(3.3)$$

Since the boundary conditions (3.3) are valid for all x_1 and t, so

$$\lambda_0 = \lambda_1 = \lambda_2 \text{ at } x_2 = 0$$

and hence we conclude from (2.10) that,

$$\left.\begin{aligned} k_0 \sin\theta_0 &= k_1 \sin\theta_1 = k_2 \sin\theta_2 \\ k_0 c_T &= k_1 c_T = k_2 c_T' . \end{aligned}\right\} \qquad \ldots(3.4)$$

Therefore, the set of boundary conditions (3.3) gives two equations for amplitudes A_1 and A_2 in terms of A_0 and is written with the aid of (3.4) as

$$\begin{bmatrix} 1 & -1 \\ 1 & T \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = A_0 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \ldots(3.5)$$

where

$$T = \frac{k_2}{k_0} \frac{R_1'}{R_1} \frac{\cos\theta_2}{\cos\theta_0} .$$

In the absence of initial stress the number of elastic moduli reduces since $Q_1 = Q_2 = \mu$ and hence, when both the media are free of initial stress and sandiness the system of equation (3.5) coincides with Achanbach (1973).

From (3.4) we find,

$$\sin \theta_2 = \frac{c'_T}{c_T} \sin \theta_0.$$

Therefore, for a given value of θ_0 we can determine the amplitude ratios of both the reflected and refracted waves for different values of all other parameters.

4. Numerical Calculation and Graphical Representation

Setting $\nu_1 = S_{11}/2Q_2$, $\nu_2 = S_{22}/2Q_1$, $\nu'_1 = S'_{11}/2Q'_2$ and $\nu'_2 = S'_{22}/2Q'_1$ and taking the values of Q_2/Q_1, Q'_2/Q'_1, Q'_1/Q_1 and ρ'/ρ as 0.87, 0.77, 1.67 and 1.87 respectively, the amplitude ratios A_1/A_0 and A_2/A_0 are calculated for different values of ν_1, ν_2, ν'_1, ν'_2, η' and θ_0. The negative values of ν_1 and ν_2 as well as ν'_1 and ν'_2 correspond to the initial compressive stresses while the positive value correspond to the initial tensile stresses. The values of η' are taken to be 1, 1.5 and 2.

We have considered the following cases :

(I) Both the media free of initial stresses and sandiness so that

$$\nu_1 = \nu_2 = 0 = \nu'_1 = \nu'_2,\ \eta = 1 = \eta',\ Q_1 = Q_2,\ Q'_1 = Q'_2 .$$

(II) The upper medium (M_2) is sandy and both the media are free of initial stresses so that $\nu_1 = \nu_2 = 0 = \nu'_1 = \nu'_2$, $\eta = 1$, $\eta' = 1.5, 2$, $Q_1 = Q_2$, $Q'_1 = Q'_2$.

(III) The upper medium is sandy and both the media are under different types of initial stress :

(i) Initial stress is compressive ;

$$\nu_1 = -0.8 = \nu'_1,\ \nu_2 = -0.4 = \nu'_2$$

(ii) Initial stress is tensile;

$$\nu_1 = 0.8 = \nu'_1,\ \nu_1 = 0.4 = \nu'_2$$

(iii) The lower medium is under compression while the upper medium is under tension;

$$\nu_1 = -0.8 = -\nu_1', \ \nu_2 = -0.4 = -\nu_2'.$$

(iv) The lower medium is under tension while the upper medium is under compression;

$$\nu_1 = 0.8 = -\nu_1', \ \nu_2 = 0.4 = -\nu_2'.$$

Besides, the values of A_1/A_0 and A_2/A_0 have been calculated for different values of ν_2 (-0.8 to 0.8 with an interval 0.2) and η' (= 1, 1.5) when $\nu_1 = \nu_1' = 0 = \nu_2'$ and

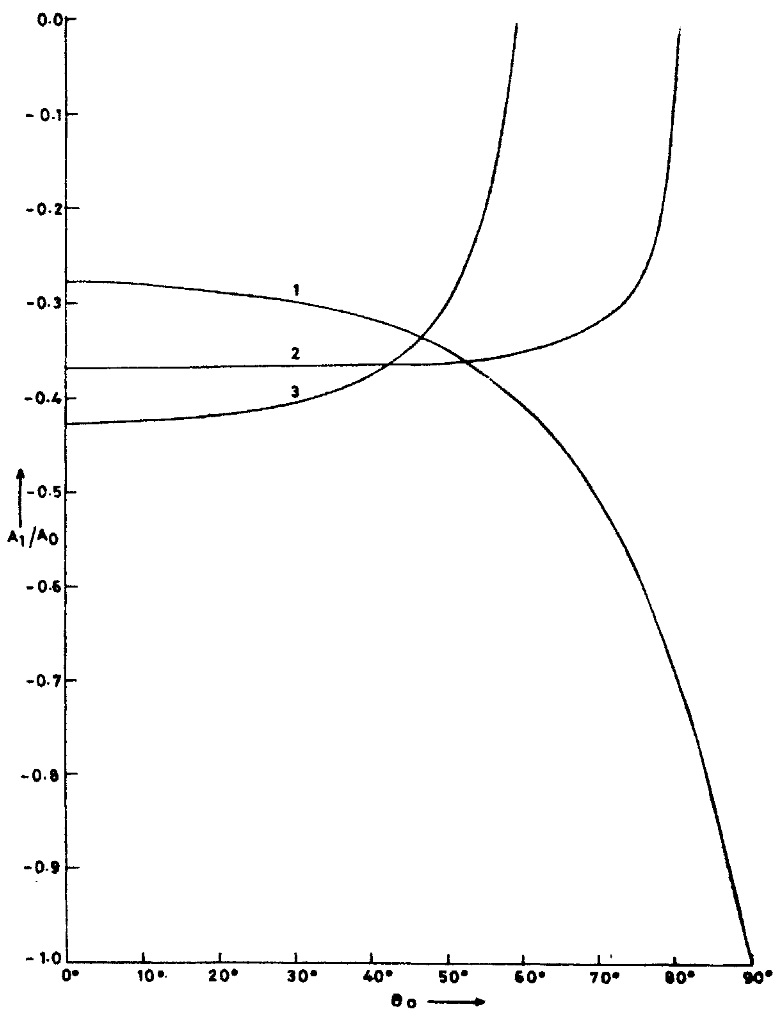

FIG. 2. Variations of A_1/A_0 with respect to θ_0, 1 — 3 : $\nu_1 = \nu_1' = 0 = \nu_2 = \nu_2'$, $\eta' = 1,\ 1.5,\ 2$.

$\theta_0 = 60°$. Also these values have been calculated for different values of ν_2' and η' when $\nu_1 = \nu_1' = 0 = \nu_2$ and $\theta_0 = 60°$.

All these calculations have been performed by the computer ICL 1901-A and these results are taken upto three decimal places.

These results are represented graphically in Figs. 2–9. In each of the Figs. 2 and 5 the curve 1 correspond to the Case I, the curves 2 and 3 correspond to the Case

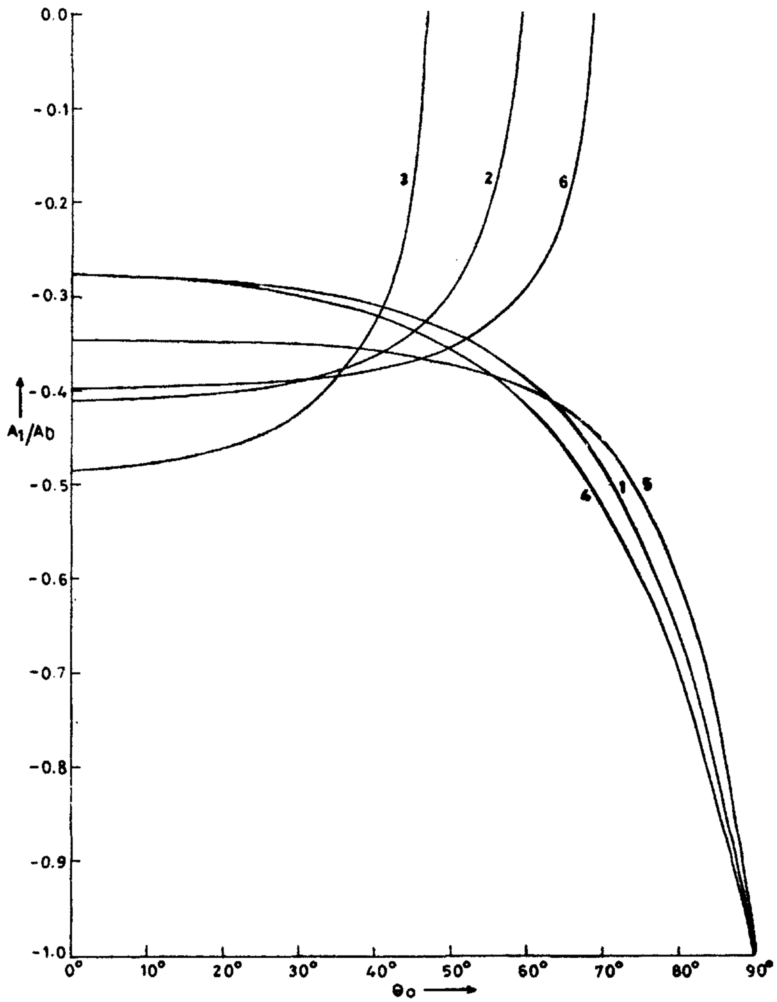

FIG. 3. Variations of A_1/A_0 with respect to θ_0, 1 — 3 : $\nu_1 = -0.8 = \nu_1'$, $\nu_2 = -0.4 = \nu_2'$, $\eta' = 1, 1.5, 2$; 4 — 6 : $\nu_1 = 0.8$ ν_1', $\nu_2 = 0.4 = \nu_2'$ $\eta' = 1, 1.5, 2$.

II. The curves 1–3 and 4–6 in each of the Figs. 3 and 6 correspond to the Case III (i) and the Case III (ii) respectively. Similarly the curves 1–3 and 4–6 in each of the Figs. 4 and 7 correspond to the Case III (iii) and the Case III (iv) respectively. Also the curves in the Figs. 8 and 9 represent the variations of the amplitudes of both the reflected and refracted waves with respect to ν_2 and ν_2' respectively for a fixed angle of incidence, viz. $\theta_0 = 60°$.

5. Discussion and Conclusion

When both the media are free of initial stress, the curves 2 and 3 in each of the Figs. 2 and 5 on comparison with the curve 1 in the same figures lead us to the con-

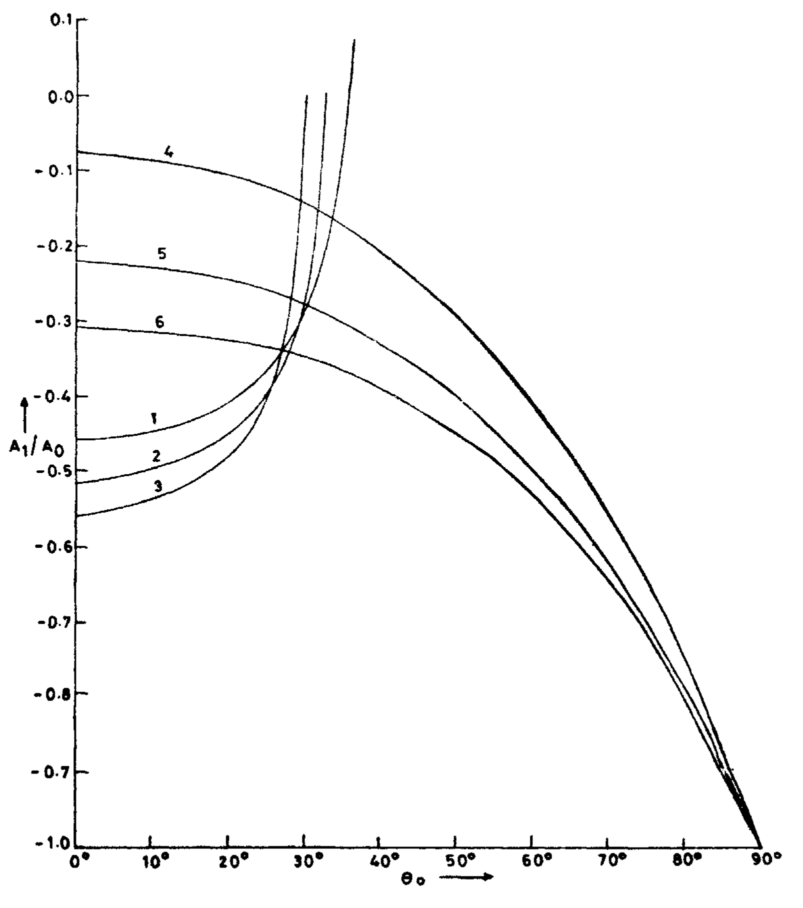

Fig. 4. Variations of A_1/A_0 with respect to θ_0 1 — 3 ; $\nu_1 = -0.8 = -\nu_1'$, $\nu_2 = -0.4 = -\nu_2'$, $\eta' = 1, 1.5, 2$: 4 — 6 : $\nu_1 - 0.8 = -\nu_1'$. $\nu_2 = -0.4 = -\nu_2'$, $\eta' = 1, 1.5, 2$.

clusion that, due to the presence of sandiness in the upper half-space, the amplitude of the reflected SH wave increases numerically up to a certain angle of incidence (i.e. $\theta_0 = 51.5°$ when $\eta' = 1.5$ and $\theta_0 = 46°$ when $\eta' = 2$) and then decreases and finally vanishes at $\theta_0 = 78.5°$ for $\eta' = 1.5$ and at $\theta_0 = 58.5°$ for $\eta' = 2$ whereas the reverse result is obtained in the case to the refracted SH wave having only greater amplitudes.

The curve 1 in the Figs. 3 and 6 on comparison with the curve 1 in Fig. 2 we observe that when both the media are under compression, the amplitude of the reflected SH wave decreases numerically for $10° < \theta_0 < 90°$ whereas the amplitude

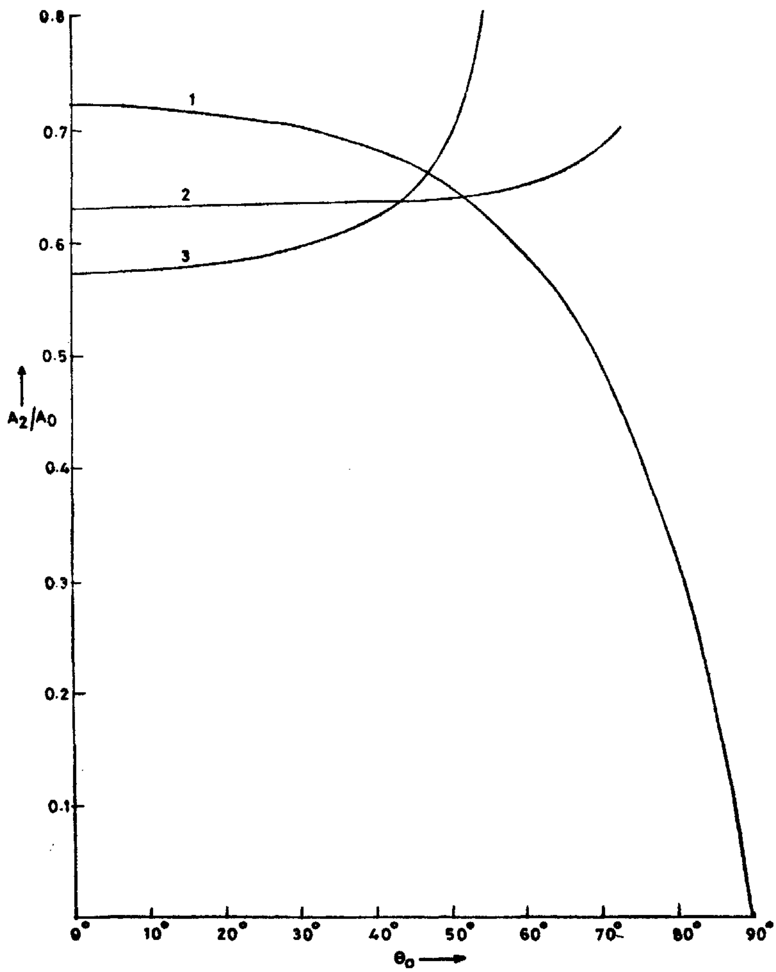

FIG. 5. Variations of A_2/A_0 with Respect to θ_0. 1 — 3 : $\nu_1 = \nu_1' = 0 = \nu_2 = \nu_2'$ $\eta' = 1, 1.5, 2$.

of the refracted SH wave increases. But when both the media are under tension one gets just the opposite results in both the reflected and refracted waves as is obvious from the curve 4 in Figs. 3 and 6 when compared with the curve 1 in Fig. 2.

Also, due to the presence of sandiness in the upper half-space, we see from the curves 1–3 Figs. 3 and 6 that when both the media are under compression, the nature of the variations of the amplitudes of both the reflected and refracted waves is the same as in the Case II which has been discussed earlier.

But that the effect of sandiness has got some pecularities when both the media are under tension can be seen from the following :

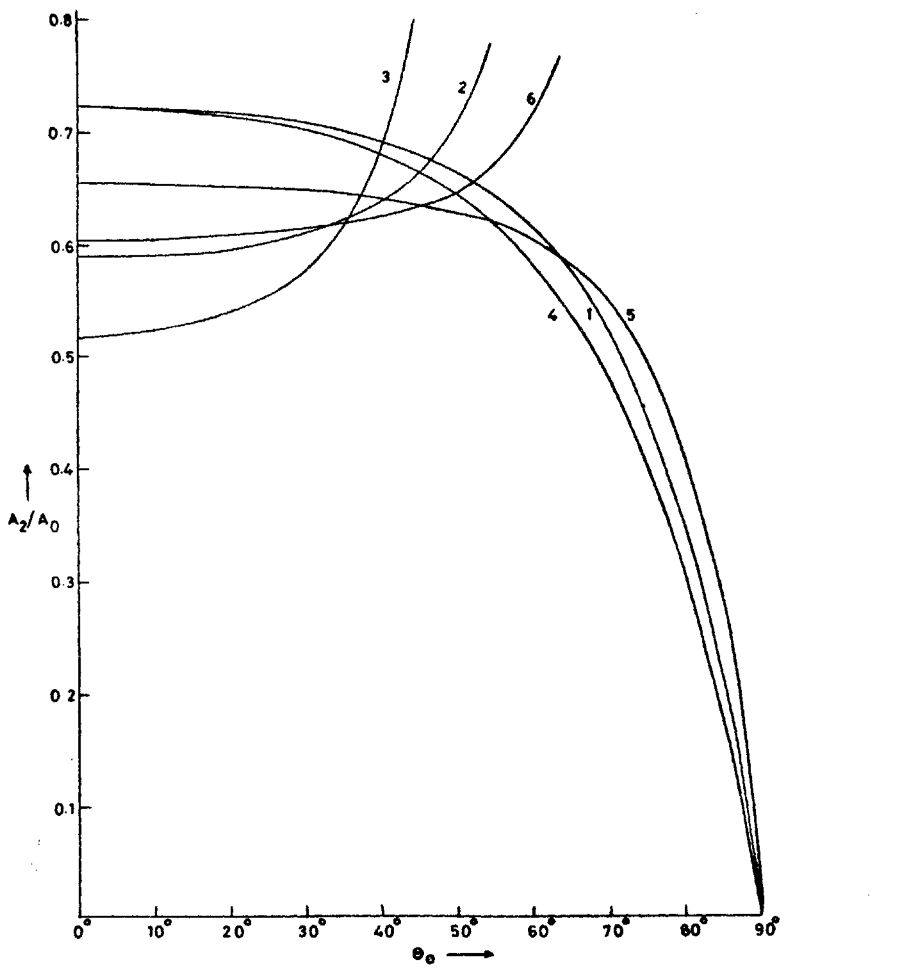

FIG. 6. Variations of A_2/A_0 with respect to θ_0, 1 — 3: $\nu_1 = -0.8 = \nu_1'$, $\nu_2 = -0.4 = \nu_2'$, $\eta' = 1, 1.5, 2$; 4 — 6 : $\nu_1 = 0.8 = \nu_1'$, $\nu_2 = 0.4 = \nu_2'$, $\eta' = 1, 1.5, 2$.

(a) for $\eta' = 1.5$, the amplitude of the reflected SH wave increases numerically up to a certain angle of incidence (i.e. $\theta_0 = 54.5°$) and then decreases and finally takes the value -1 at $\theta_0 = 90°$.

(b) for $\eta' = 2$, the nature of the variations of the amplitude of the reflected SH wave is the same but it takes the value of 0 at $\theta_0 = 68.5°$.

In the case of the refracted SH wave, the nature of the variations of the amplitude is just opposite as is obvious from the curves 4–6 in Figs. 3 and 6.

It is interesting to note from the curves 1–3 in Figs. 4 and 7 that when the upper medium is under tension and the lower medium is under compression, no real ampli-

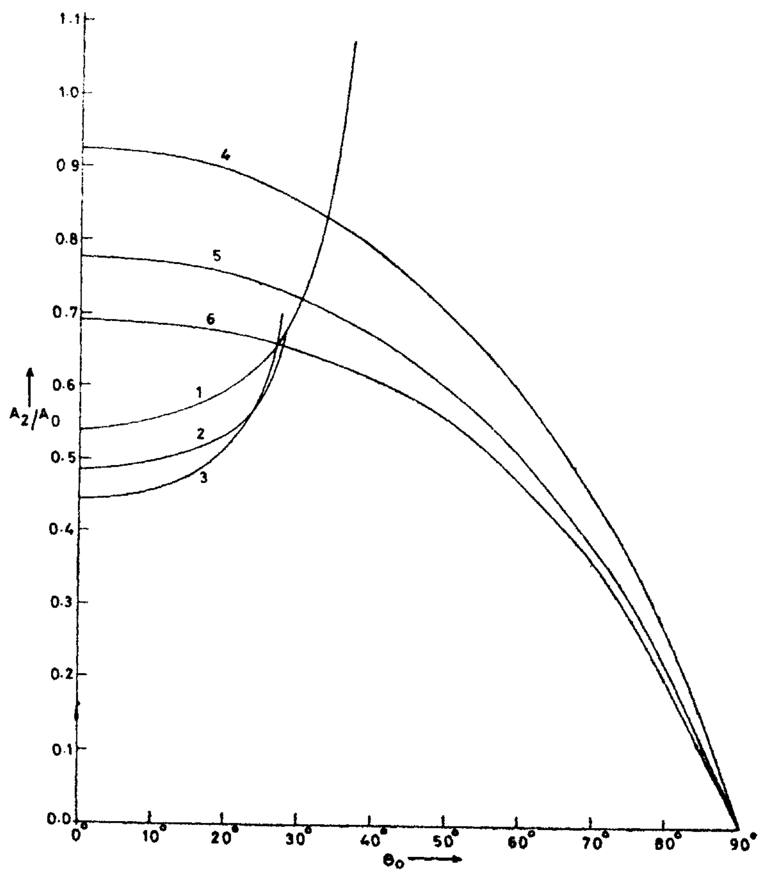

FIG. 7. Variations of A_2/A_0 with respect to θ_0. 1 — 3 : $\nu_1 = -0.8 = -\nu_1'$, $\nu_2 = -0.4 = -\nu_2'$, $\eta' = 1, 1.5, 2$. 4 — 6 : $\nu_1 = 0.8 = -\nu_1'$, $\nu_2 = -0.4 = -\nu_2'$ $\eta' = 1, 1.5, 2$.

tudes of both the reflected and refracted SH waves exist for $\theta_0 > 40.5° \pm 4.5°$ when $\eta' = 1$ and for $\theta_0 > 32.2° \pm 4.5°$ when $\eta' = 1.5$ and 2. The effect of sandiness on the amplitudes of the reflected SH wave is to increase it numerically whereas it decreases the amplitude of the refracted SH wave.

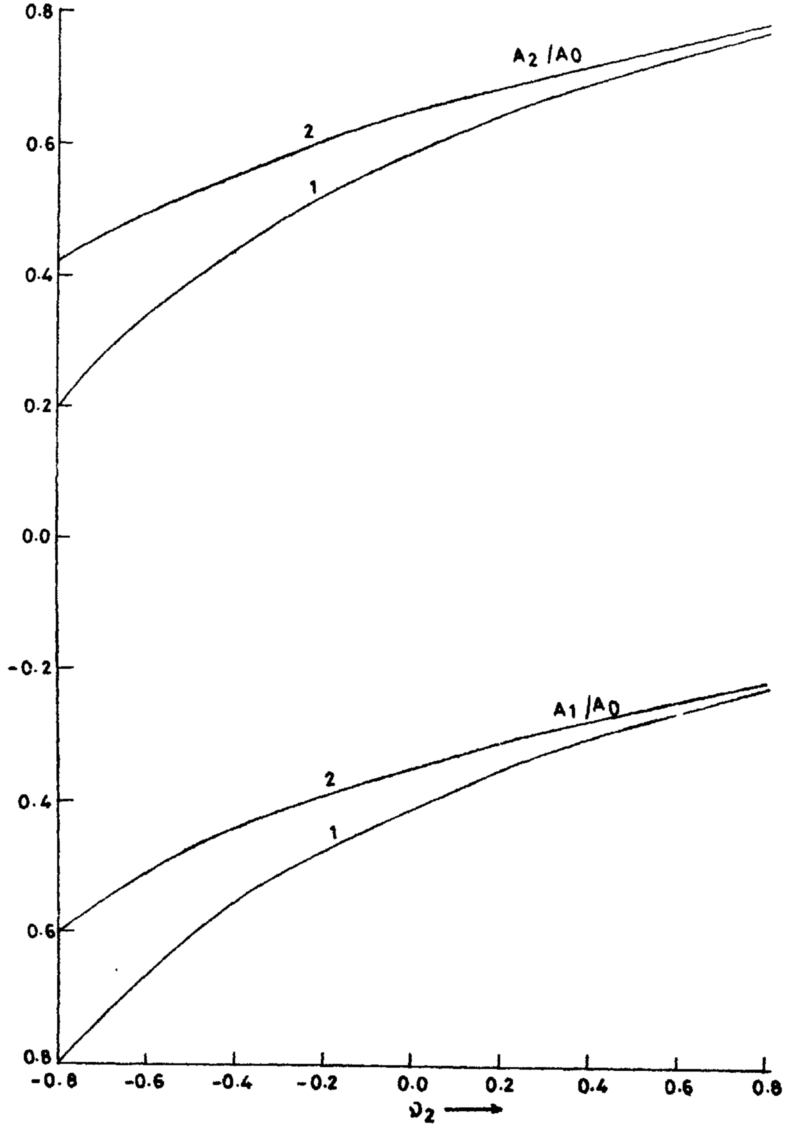

FIG. 8. Variations of A_1/A_0 and A_2/A_0 with respect to ν_2 when $\nu_1 = \nu_1' = 0 = \nu_2'$ and $\theta_0 = 60°$ 1 — 2 : $\eta' = 1, 1.5$.

But when the upper medium is under compression and the lower medium is under tension, the nature of the variations of the amplitudes of both the reflected and refracted waves is the same as in the Case I having only greater amplitudes for refracted SH wave and less amplitudes for reflected SH wave. Due to the presence of sandiness in the upper half-space, the amplitude of the reflected SH wave increases numerically whereas that of the refracted SH wave decreases which is obvious from the curves 4-6 in Figs. 4 and 7.

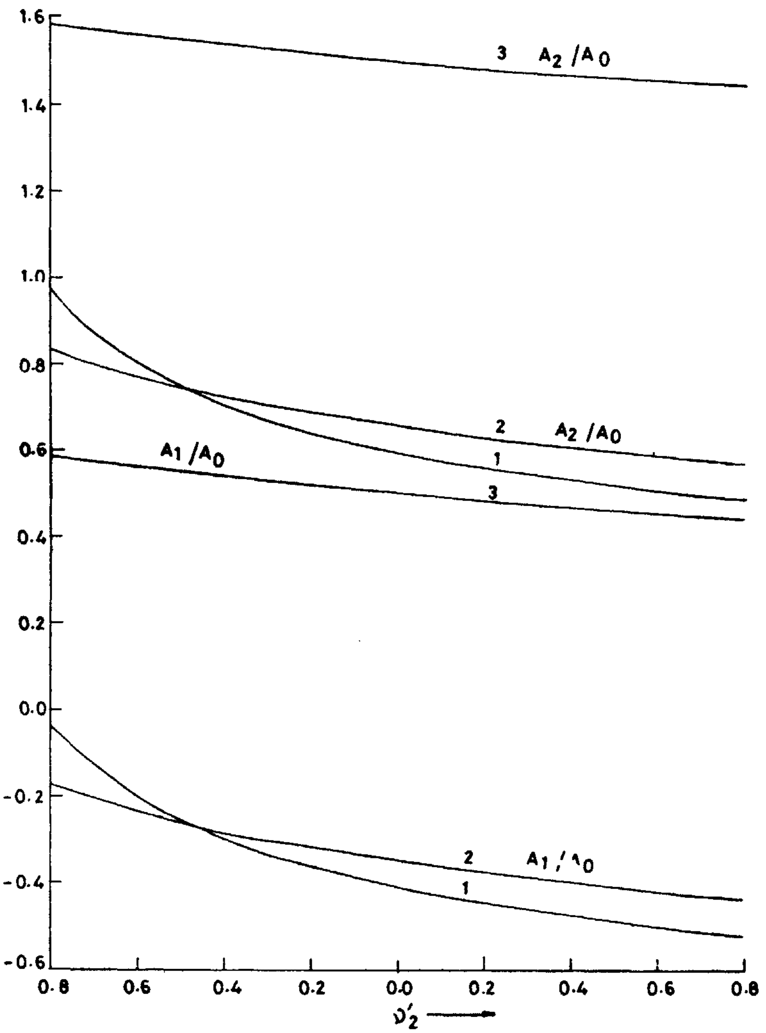

FIG. 9. Variations of A_1/A_0 and A_2/A_0 with respect to ν_2', when $\nu_1 = \nu_1' = 0 = \nu_2$ and $\theta_0 = 60°$. 1 — 3 - $\eta' = 1, 1.5, 2$.

When the upper medium is free of initial stress and the lower medium possesses the initial stress in the x_2 direction only, we see from the Figs. 8 that for a fixed angle of incidence viz. $\theta_0 = 60°$, the amplitude of the reflected SH wave increases or decreases numerically according as the vertical initial stress is compressive or tensile while the amplitude of the refracted SH wave increases or decreases according as the initial stress is tensile or compressive. Due to the sandiness the nature of the variation of the amplitudes with respect to initial stress is the same but it decreases the amplitude of the reflected SH wave and increases that of the refracted SH wave. It is noted that no real amplitude exists here for $\eta' = 2$.

When the lower medium is free of initial stress and the upper medium possesses the initial stress in the x_2-direction only, we observe from the Fig. 9 that for the same angle of incidence viz. $\theta_0 = 60°$, the amplitude of the reflected SH wave increases or decreases accordingly as the initial stress is tensile or compressive whereas the reverse results is observed in the case of the refracted SH waves. The presence of sandiness shows that, for $\eta' = 1.5$ the amplitude of the reflected SH waves increases numerically and that of the refracted SH wave decreases but for $\eta' = 2$ the amplitude of both the reflected and refracted SH waves increases.

Finally, from the amplitudes of both the reflected and refracted waves which are represented in the Figs. 2–9, we conclude that the presence of initial stress and the sandiness has considerable effect on the reflection and refraction phenomena of elastic waves in addition to the angles of incidences.

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