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A NOTE ON GENERALIZED SYLVESTER POLYNOMIALS

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In this paper three different classes of generating functions and various elegant summation formulae for generalized Sylvester polynomials, studied by Agarwal and Manocha (1980), are obtained. Specializing a result of Srivastava (1980), a multivariable case of a trilateral generating relation due to Agarwal and Manocha (1980) is also given.

1. INTRODUCTION

Agarwal *et al.* (1980, 1982) studied the generalized Sylvester polynomials by means of the generating relation [cf. (1.2) below]

$$\sum_{n=0}^{\infty} f_n(x; a)\, t^n = (1 - t)^{-x}\, e^{axt}. \qquad \ldots(1.1)$$

The result obtained by them for these polynomials include hypergeometric representation, some linear at trilateral generating functions and a differential operator representation. In the present paper three different classes of generating functions and various elegant summation formulae for these polynomials are obtained [see also (1.3) below]. In the last section, by suitably specializing Corollary 10 of Srivastava (1980, p. 232) a multivariable extension of a trilateral generating relation due to Agarwal and Manocha (1980) is also given.

It is interesting to mention here that if we consider the generating relation [see, fon instance, Erdélyi *et al.* (1953), p. 189]

$$\sum_{n=0}^{\infty} L_m^{(\alpha-n)}(x)\, t^n = (1 + t)^{\alpha}\, e^{-xt} \qquad \ldots(1.2)$$

we find that the generalized Sylvester polynomials have the following relationship with the modified Laguerre polynomials :

$$f_n(x; a) = (-1)^n\, L_n^{(x-n-)}(ax) \qquad \ldots(1.3)$$

which indeed was pointed out by the referee.

In view of the relationship (1.3), the results obtained in this paper for the polynomials $f_n(x; a)$ can be readily deduced from the corresponding known results for the modified Laguerre polynomials. Proofs are, therefore, omitted.

2. Classes of Generating Functions

In this section we first state Theorem 2.1. Let b and c be arbitrary constants. Then the polynomials $f_n(x; a)$ defined by (1.1) above satisfy the following generating relations

$$\sum_{n=0}^{\infty} f_n(x; a+bn)\, t^n = \frac{e^{au}}{1-bu}\left(1-\frac{u}{x}\right), \quad u = xte^{bu} \qquad \ldots(2.1)$$

$$\sum_{n=0}^{\infty} f_n(x+cn; a)\, t^n = \frac{(1-v)^{-x}\, e^{axv}}{1-v[c(1-v)^{-1}+ac]} \qquad \ldots(2.2)$$

$$v = t(1-v)^{-c}\, e^{acv}$$

and

$$\sum_{n=0}^{\infty} f_n\left(x+cn; \frac{a+bn}{x+cn}\right) t^n = \frac{(1-w)^{-x}\, e^{aw}}{1-w[c(1-w)^{-1}+b]} \qquad \ldots(2.3)$$

$$w = t(1-w)^{-c}\, e^{bw}.$$

Formulas (2.1), (2.2) and (2.3) are immediate consequences of Carlitz's Theorem (1977). [For generalizations Carlitz's theorem see Srivastava (1979)].

3. Summation Formulae

The following summation formulae are easily derivable from known results in view of the relationship (1.3):

$$f_n(x; a) = \sum_{r=0}^{n} \frac{\{x(a-b)\}^r f_{n-r}(x; b)}{r!} \qquad \ldots(3.1)$$

$$f_n(x+y; a) = \sum_{r=0}^{n} f_{n-r}(x; a)\, f_r(y; a) \qquad \ldots(3.2)$$

$$f_n(x; a+b) = \sum_{r=0}^{n} \frac{(bx)^r f_{n-r}(x; a)}{r!} \qquad \ldots(3.3)$$

$$\sum_{r=0}^{n} f_{n-r}(x; a+b)\, f_r(y; a+b) = \sum_{r=0}^{n} \frac{\{b(x+y)\}^r f_{n-r}(x+y; a)}{r!} \qquad \ldots(3.4)$$

and

$$(n + 1) f_{n+1} (x; a) = x \left[a f_n (x) + \sum_{r=0}^{n} f_{n-r} (x) \right]. \qquad \ldots(3.5)$$

It may be pointed out here that the following summation formula has been given earlier [see, for instance, Agarwal *et al.* (1982), eqn. (3.16), p. 243]:

$$f_n (x; ab) = \sum_{r=0}^{n} \frac{\{ax (b - 1)\}^{n-r}}{(n - r)!} f_k (x; a). \qquad \ldots(3.6)$$

4. Multilateral Generating Function

By specializing Corollary 10 (p. 232) of Srivastava (1980) we give the following multivariable case of a trilateral generating function due to Agarwal and Manocha (1980):

Theorem 4.1 [cf. Srivastava (1980, p. 232, Corollary 10]—For the polynomials $f_n (x; a)$ defined by (1.1), let

$$H_{p,q}^{\mu,m} [x; a; y_1, \ldots, y_s; t] = \sum_{n=0}^{\infty} a_{n,\mu} f_{nq+m} (x; a) g_{np+\mu} (y_1, \ldots, y_s) t^n \qquad \ldots(4.1)$$

where p and q are positive integers, μ an arbitrary complex number, and $g_\mu (y_1 \ldots, y_s)$ s a non-vanishing function of s variables $y_1, \ldots, y_s$, $s \geqslant 1$.
Then

$$\sum_{n=0}^{\infty} f_{m+n} (x; a) \, \Omega_{n,p,q}^{\mu,m} (y_1, y_2, \ldots, y_s; z) t^n, \; n \geqslant 0$$

$$= (-t)^{-x-m} e^{axt} H_{p,q}^{\mu,m} \left[[x; a (1 - t); y_1, \ldots, y_s; z \left(\frac{t}{1-t} \right)^q \right] \qquad \ldots(4.2)$$

where

$$\Omega_{n,p,q}^{\mu,m} (y_1, \ldots, y_s; z) = \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_{k,\mu} \, g_{kp+\mu} (y_1, \ldots, y_s) z^k. \qquad \ldots(4.3)$$

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