Indian J. pure appl. Math., 15(1) : 9–14, January 1984

GENERALIZED SCHWARZSCHILD INTERIOR SOLUTION

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(*Received* 6 *November* 1982; *after revision* 29 *July* 1983)

A spherically symmetric metric in the conformally flat form is investigated subject to perfect fluid conditions. The cases with variable and constant energy-density are studied thoroughly. Consequently a non-static generalization of Schwarzschild's interior solution is obtained. Also the Schwarzschild's interior solution is determined as an envelope of one parameter family of de-Sitter Universes.

1. INTRODUCTION

P. C. Vaidya (1968) worked out a class of non-static spherically symmetric solutions representing perfect fluid distribution and possessing the property that the four dimensional stream lines are orthogonal to hyper surface ρ (energy-density) = constant. In the said distribution $\partial\rho/\partial r$ and $\partial\rho/\partial t$ both were non-zero with an exception of Schwarzschild's interior solution where the energy-density is merely a positive constant. Later on the above class of solutions was shown to be conformally flat and of class one (1971), provided the energy-density is non zero (Barnes 1974).

In this article authors have attempted the problem starting with a spherically symmetric metric conformal to a Minkowaskian metric in the spherical co-ordinates, and obtained fluid distributions subject to various restrictions on energy-density such as

$$\text{(i)}\ \ \frac{\partial\rho}{\partial r}\neq 0,\ \frac{\partial\rho}{\partial t}\neq 0 \qquad \text{(ii)}\ \ \frac{\partial\rho}{\partial r}=0,\ \ \frac{\partial\rho}{\partial t}\neq 0$$

$$\text{(iii)}\ \frac{\partial\rho}{\partial r}\neq 0,\ \frac{\partial\rho}{\partial t}=0 \quad \text{(iv)}\ \ \rho=\text{constt.}$$

The solution in case (i) is similar to that of Vaidya but more explicit than the later. The case (ii) represents the non-static generalization of a special case of Schwarzschild's interior solution. Solution does not exist for case (iii). In case (iv) Schwarzschild's interior solution is obtained as an envelope of one parameter family of de-Sitter Universes. An empty field is also deduced. However an envelope of two parameter family of de-Sitter Universes gives rise to an electrovac solution.

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2. Density Varying with Position and Time

A spherically symmetric metric in conformally flat form can be expressed as,

$$ds^2 = A^{-2}(r, t)\,[-dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2]. \qquad \ldots(2.1)$$

Further the Einstein field equations for a perfect fluid distribution can be written as

$$R^i_j - \frac{1}{2} R\delta^i_j = -8\pi T^i_j = -(a+b)\,v^i v_j + b\,\delta^i_j \qquad \ldots(2.2)$$

where $a = 8\pi\rho$, $b = 8\pi p$, $v^i v_i = 1$,

ρ, p and v_i being energy-density, pressure and flow vector respectively.

Equation (2.2) for the metric (2.1) gives rise to

$$8\pi\ T^1_1 = -3(A'^2 - \dot{A}^2) - 2A\ \ddot{A} + \frac{4}{r} AA' = (a+b)\,v^1 v_1 - b \qquad \ldots(2.3)$$

$$8\pi\ T^2_2 = 8\pi\ T^3_3 = -3(A'^2 - \dot{A}^2) + 2A(A'' - \ddot{A}) + \frac{2}{r} AA' = -b \qquad \ldots(2.4)$$

$$8\pi\ T^4_4 = -3(A'^2 - \dot{A}^2) + 2AA'' + \frac{4}{r} AA' = (a+b)\,v^4 v_4 - b \qquad \ldots(2.5)$$

$$8\pi\ T^4_1 = -8\pi\ T^1_4 = 2A\dot{A}' = -(a+b)\,v^1 v_4 = (a+b)\,v^4 v_1 \qquad \ldots(2.6)$$

$$v^2 = v_2 = v^3 = v_3 = 0 \text{ and } v^1 v_1 + v^4 v_4 = 1. \qquad \ldots(2.7)$$

The prime and dot denote partial differentiation with respect to r and t respectively.

Eliminating a, b and v_i among (2.3) to (2.7) we get

$$\left(\ddot{A} + \frac{A'}{r}\right)\left(A'' - \frac{A'}{r}\right) = \dot{A}'^2. \qquad \ldots(2.8)$$

Equation (2.8) is not easily solvable for A because of its highly non-linear character. Therefore an alternative method is used in solving the above problem.

Adding (2.3) and (2.5) and making use of (2.4) and (2.7) we get

$$a = \frac{6AA'}{r} - 3(A'^2 - \dot{A}^2). \qquad \ldots(2.9)$$

Moreover an easy analysis of (2.3)–(2.9) gives rise to

$$\ddot{\;} + \frac{A'}{r} = \frac{v_4}{v_1}\dot{A}',\ A'' - \frac{A'}{r} = \frac{v_1}{v_4}\dot{A}' \qquad \ldots(2.10)$$

(2.9) and (2.10) further produce,

$$\frac{a_{,r}}{a_{,t}} = \frac{v_1}{v_4}, \text{ where } a_{,r} \equiv \frac{\partial a}{\partial r},\ a_{,t} \equiv \frac{\partial a}{\partial t} \qquad \ldots(2.11)$$

which implies the orthogonality of flow lines to the hypersurfaces ρ (energy-density) = constt. and agrees with the results of Vaidya (1968), Barnes (1974), Gupta (1976), Rao (1972).

Equation (2.10) together with (2.11) gives

$$\ddot{A} - \frac{a_{,t}}{a_{,r}}\dot{A}' = -\frac{A'}{r} \text{ and } A'' - \frac{a_{,r}}{a_{,t}}\dot{A}' = \frac{A'}{r} \qquad \ldots(2.12)$$

where $a_{,r}$ and $a_{,t}$ are non zero.

(2.12) on integration gives

$$\dot{A} = -t f(a) + g(a) \text{ and } A' = rf(a) \qquad \ldots(2.13)$$

where f and g are arbitrary functions of a. Consistency of the two relations in (2.13) demands

$$a_{,r}(\bar{g} - t\bar{f}) = a_{,t}\, r\bar{f} \qquad \ldots(2.14)$$

where bar indicates differentiation with respect to a. (2.14) on integration gives,

$$(r^2 - t^2)\cdot\bar{f} + 2\bar{g}\cdot t = 2H(a) \qquad \ldots(2.15)$$

where $H(a)$ is an arbitrary function of a.

On substituting the values of $\dot{A}$ and A' from (2.13) into (2.9), we get

$$A = \frac{1}{2f}\left[\frac{a}{3} + (r^2 - t^2)\cdot f^2 + 2tgf - g^2\right]. \qquad \ldots(2.16)$$

Now the consistency of (2.13) and (2.16) on making use of (2.15) requires

$$H(a) = \frac{1}{2}\frac{d}{da}\left(\frac{g^2}{f} - \frac{a}{3f}\right). \qquad \ldots(2.17)$$

Consequently (2.15) together with (2.17) assumes the form

$$(r^2 - t^2)\bar{f} + 2\bar{g}t = \frac{d}{da}\left(\frac{g^2}{f} - \frac{a}{3f}\right). \qquad \ldots(2.18)$$

Therefore the solution to the problem with variable energy-density (i.e. $a_{,r} \neq 0$, $a_{,t} \neq 0$) is described by the metric (2.1) along with (2.16) and (2.18). The pressure and flow-vector can easily be written by inserting 'A' from (2.16) into the proper expressions. The above solution is a transform of Vaidya's solution (1968). However the present solution possesses a plus point. In the present case the conformal factor A^{-2} is expressible in terms of r, t and energy-density, while the energy-density satisfies a simple equation which does not involve derivatives of the later. In case of Vaidya's solution the metric potentials were functions of same argument but energy-density was to satisfy a first order differential equation.

Many special cases can be deduced from (2.16) by assigning various values to $f(a)$ and $g(a)$ such as :

(i) $f = 2\alpha g$, when substituted in (2. 18) implies that

$$a = a\,[\alpha(r^2 - t^2) + \beta t + \gamma]$$

α, β and γ being arbitrary constants.

Now with the help of (2.16) one can easily assert that A must also be of the same form i.e.,

$$A = A\,[\alpha\,(r^2 - t^2) + \beta t + \gamma]. \qquad \ldots(2.19)$$

It is worthnoting here that the solution due to Singh and Sattar (1974) is a special case of (2.19).

Further (2.19) is seen to represent the most general solution subject to the condition $A = A\,(a)$.

(ii) g = constt., when substituted in (2.18) implies energy-density to be a function of $(r^2 - t^2)$ and therefore (2.16) assumes the form

$$A = \phi\,(r^2 - t^2) + \beta t. \qquad \ldots(2.20)$$

(2.20) represents a non-static generalization of Schwarzschild's interior solution. Its special case

$$A = \alpha\,\sqrt{(r^2 - t^2)} + \beta t \qquad \ldots(2.21)$$

where α and β are arbitrary constants, will be shown to describe Schwarzschild's interior solution in the next section.

In the preceding discussion energy-density was essentially a function of r and t both. Now we shall analyse the cases (i) energy-density as a function of r only (ii) as a function of t only.

Making use of (2.8) and (2.9) we get

$$a_{,r}\,\dot{A}' = a_{,t}\left(A'' - \frac{A'}{r}\right). \qquad \ldots(2.22)$$

An analysis of (2.22) and (2.8) gives $a_{,r} = 0$ and $a_{,t} \neq 0$ need $A' = 0$ and $a_{,t} = 0$ and $a_{,r} \neq 0$ also need $\dot{A}' = 0$.

While $\dot{A}' \neq 0$ implies the constancy of energy-density. Therefore the condition $\dot{A}' = 0$ is necessary for the cases $a_{,t} = 0$ or $a_{,r} = 0$. Let us investigate the relation $\dot{A}' = 0$.

The said condition and the equation (2.8) give rise to the solution

$$A = \alpha\,r^2 + K\,(t) \qquad \ldots(2.23)$$

where α and K are arbitrary constant and arbitrary function of t respectively.

The expressions for density and pressure for (2.23) are given by

$$a = 12\alpha K + 3\dot{K}^2$$

$$b = -\,3\dot{K}^2 + 2\ddot{K}\,(\alpha r^2 + K) + 4\alpha\,(\alpha r^2 - 2K)$$

which represents an expanding universe model. Elsewhere we have proved this to be a non-static generalization of a special case of Schwarzschild's interior solution (1969). It is worthnoting here that density depends upon time only and a case with $a_{,t} = 0$ reduces to a case of constant density. Hence the case $a_{,t} = 0$, $a_{,r} \neq 0$ does not exist.

3. Constant Energy-Density

With the help of eqns. (2.8) and (2.9) the expression of A for a = constt. is similar to that given in (2.16) with f and g to be arbitrary constants. Using eqns. (2.4) and (2.9) we get

$$a + b = 0 \qquad \ldots(3.1)$$

which indicates that the solution with constant energy-density, so obtained. is nothing but a de-Sitter's solution.

Let us study the envelopes of the family of de-Sitters' Universes, considering it as (i) one parameter family (with parameter f or g) and as (ii) two parameter family (with parameters f and g).

(a) With respect to parameter f alone : Differentiating (2.16) partially with respect to f and equating it to zero,

$$\frac{\partial A}{\partial f} = -\frac{a}{6f^2} + \frac{1}{2}(r^2 - t^2) + \frac{g}{2f^2} = 0. \qquad \ldots(3.2)$$

Eliminating f between (2.16) and (3.2), we have,

$$A = \sqrt{\left(g^2 - \frac{a}{3}\right)(t^2 - r^2)} + g \cdot t. \qquad \ldots(3.3)$$

It is worth mentioning here that (2.1) with (3.3) can be transformed to the standard canonical metric of Schwarzschild interior solution

$$ds^2 = -\left(1 - \frac{a\bar{r}^2}{3}\right) d\bar{r}^2 - \bar{r}^2 d\Omega^2 + \left[\sqrt{\left(g^2 - \frac{a}{3}\right)} + g\left(1 - \frac{a}{3}\bar{r}^2\right)^{1/2}\right]^2 dT^2 \qquad \ldots(3.4)$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$

by means of a transformation

$$\bar{r} = \frac{r}{A}, \; T = \frac{3}{2a} \log(t^2 - r^2).$$

However when energy-density is zero; i.e. $a = 0$ (empty field) (2.1) with (3.3) can be transformed to

$$ds^2 = -d\bar{r}^2 - \bar{r}^2 d\Omega^2 + (1 - \bar{r}^2 g^2)^2 dT^2 \qquad \ldots(3.5)$$

through the transformation

$$\bar{r} = \frac{r}{A}, \; T = \frac{1}{4g} \log(t^2 - r^2)$$

which is the static sub-case of the solution obtained by Barnes (1974). However in (3.5) g can be made a function of T and still it is a solution of (2.2) with $a = 0$ (Pachlaner 1974).

(b) With respect to parameter g only

$$\frac{\partial A}{\partial g} = \frac{1}{2f}[2tf - 2g] = 0. \qquad \ldots(3.6)$$

Eliminating g between (2.16) and (3.6) we get

$$A = \frac{a}{6f} + \frac{1}{2} fr^2. \quad \ldots(3.7)$$

(3.7) is a static sub-case of (2.23).

(c) With respect to two parameters f and g.

Eliminating f and g among (2.16), (3.2) and (3.6) we get

$$A = r\sqrt{a/3}. \quad \ldots(3.8)$$

(2.1) with (3.8) represents a solution with constant curvature and scalar curvature zero used in the theory of the atom based on wave geometry (1962). Also (3.8) is said to represent an electrovac solution (1981).

4. Conclusion

The use of spherically symmetric metric in conformally flat form made it possible to study the conformally flat perfect fluid in a very exhaustive manner and to work out the solutions explicitly.

Acknowledgement

The authors wish to express their indebtedness and sincere thanks to Professor S. N. Pandey, University of Roorkee for valuable discussions regarding the problem. The authors are thankful to the referee for his comments and suggestions for the improvement of the paper.

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