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ON SOME FIXED POINT THEOREMS

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In this note we prove, mainly, the existence of a unique fixed point for a self map on a complete metric space and the existence of a unique common fixed point for a pair of self maps on a compact metric space. The former critically examines a result of Ciric (1981) and the later extends a result of Fisher (1980).

Meir and Keeler (1969) proved the existence of a unique fixed point for a self map T on a complete metric space (X, d) satisfying the following condition :

Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon \leqslant d(x, y) < \epsilon + \delta \text{ implies } d(Tx, Ty) < \epsilon. \qquad \ldots(1)$$

Clearly (1) implies T is contractive, i.e.,

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X \text{ with } x \neq y \qquad \ldots(2)$$

and hence continuous.

Recently, Ciric (1981) proved the following :

Theorem A—Let (X, d) be a complete metric space and T a self map on X satisfying the following condition :

Given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\epsilon < d(x, y) < \epsilon + \delta \text{ implies } d(Tx, Ty) \leqslant \epsilon. \qquad \ldots(3)$$

Then T has a unique fixed point, say, $z \in X$ and for any $x \in X$, $T^n x \to z$.

He also observed that (1) and (3) are not equivalent.

However, we observe that Theorem A is not valid and T need not be contractive in view of the following :

Example—Let $X = \{0, 1\}$ with the usual metric. Let $T : X \to X$ be given by $T0 = 1$, $T1 = 0$. Then all the conditions of Theorem A are satisfied with

$$\delta(\epsilon) = \begin{cases} 1-\epsilon, & 0 < \epsilon < 1 \\ K(> 0), & \epsilon \geqslant 1. \end{cases}$$

Now we give a probable modification of Theoren A.

Theorem B—Assume all the conditions of Theorem A and further assume.

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$$d(Tx, Ty) < \text{Max}\left\{d(x, y), \frac{1}{2}\left[d(x, Tx) + d(y, Ty)\right], \frac{1}{2}\left[d(x, Ty) + d(y, Tx)\right]\right\} \quad \text{for } x \neq y. \qquad \ldots(4)$$

Then the conclusion of Theorem A holds.

PROOF : If T has a fixed point then by (4), it must be unique. Let $x_0 \in X$, define $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$.

Suppose $x_{n+1} = x_n$ for some n. Then there is nothing to prove. Assume $x_{n+1} \neq x_n$ for all n.

By (4), $\{d(x_n, x_{n+1})\}$ is a strictly decreasing sequence of nonnegative real numbers and hence converges to a nonnegative real number, say, r. Hence we have

$$d(x_n, x_{n+1}) > r \text{ for } n = 0, 1, 2, \ldots . \qquad \ldots(5)$$

Suppose $r > 0$. Then there exists a $\delta = \delta(\epsilon)$ and a positive integer k such that $r < d(x_k, x_{k+1}) < r + \delta$. Hence by (3), we have $d(x_{k+1}, x_{k+2}) = d(Tx_k, Tx_{k+1}) \leqslant r$, a contradiction to (5).

Hence $r = 0$. Thus $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.

Claim : $\{x_n\}$ is Cauchy.

Suppose not. Then there exists an $\epsilon > 0$ such that for each positive integer N there exist integers m, n with $m > n > N$ such that $d(x_m, x_n) \geqslant 2\epsilon$.

Choose $\delta > 0$ with $\delta < \epsilon$ for which (3) is satisfied.

Since $\{d(x_n, x_{n+1})\} \downarrow 0$ there exists a positive integer $N = N(\delta)$ such that

$$d(x_i, x_{i+1}) \leqslant \frac{\delta}{6} \text{ for all } i \geqslant N.$$

With this choice of N, pick m, n with $m > n > N$ such that

$$d(x_m, x_n) \geqslant 2\epsilon > \epsilon + \delta. \qquad \ldots(6)$$

By (6), $m-n > 6$, otherwise

$$d(x_m, x_n) \leqslant \sum_{i=0}^{5} d(x_{n+i}, x_{n+i+1})$$

$$\leqslant \frac{5}{6}\delta < \delta, \text{ a contradiction.}$$

Suppose

$$d(x_n, x_{m-1}) \leqslant \epsilon + \frac{1}{3}\delta.$$

Then

$$d(x_n, x_m) \leqslant d(x_n, x_{m-1}) + d(x_{m-1}, x_m) \leqslant \epsilon + \frac{\delta}{3} + \frac{\delta}{6} < \epsilon + \delta$$

a contradiction. Hence

$$d(x_n, x_{m-1}) > \epsilon + \frac{1}{3}\delta.$$

Similarly, suppose

$$d(x_n, x_{m-2}) \leqslant \epsilon + \frac{1}{3}\delta.$$

Then

$$d(x_n, x_m) \leqslant d(x_n, x_{m-2}) + d(x_{m-2}, x_{m-1}) + d(x_{m-1}, x_m) \leqslant \epsilon + \frac{1}{3}\delta + \frac{1}{6}\delta + \frac{1}{6}\delta < \epsilon + \delta, \text{ a contradiction.}$$

Thus there exists a smallest integer $j \in (n, m)$ with

$$d(x_n, x_j) > \epsilon + \frac{1}{3}\delta.$$

$$d(x_n, x_j) \leqslant d(x_n, x_{j-1}) + d(x_{j-1}, x_j) \leqslant \epsilon + \frac{1}{3}\delta + \frac{1}{6}\delta < \epsilon + \frac{2}{3}\delta.$$

Thus there exists a $j \in (n, m)$ such that

$$\epsilon + \frac{1}{3}\delta < d(x_n, x_j) < \epsilon + \frac{2}{3}\delta. \qquad \ldots(7)$$

Hence by (3), $d(x_{n+1}, x_{j+1}) = d(Tx_n, Tx_j) \leqslant \epsilon$.
Then

$$d(x_n, x_j) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{j+1}) + d(x_{j+1}, x_j)$$
$$\leqslant \frac{1}{6}\delta + \epsilon + \frac{1}{6}\delta = \epsilon + \frac{1}{3}\delta, \text{ a contradiction to (7).}$$

Hence $\{x_n\}$ is Cauchy.

Since (X, d) is complete, there exists a $z \in X$ such that $x_n \to z$ as $n \to \infty$. Clearly $x_n \neq z$ for infinitely many n. We can as well assume that $x_n \neq z$ for all n. Then by (4),

$$d(Tx_n, Tz) < \text{Max} \left\{ d(x_n, z), \frac{1}{2}\left[d(x_n, Tx_n) + d(z, Tz)\right], \frac{1}{2}\left[d(x_n, Tz) + d(z, Tx_n)\right] \right\}.$$

Letting $n \to \infty$, we have

$$d(z, Tz) \leqslant \frac{1}{2} d(z, Tz) \text{ so that } Tz = z.$$

This completes the proof.

Finally we prove a fixed point theorem in a compact metric space which improves the result of Fisher (1980).

Theorem—Let S and T be two self maps on a compact metric space (X, d) satisfying

(I) $$d(Sx, Ty) < \text{Max} \left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2}\left[d(x, Ty) + d(y, Sx)\right] \right\} \text{ for all } x, y \in X \text{ with } Sx \neq Ty.$$

Also assume that either S or T is continuous. Then S and T have a unique common fixed point, say, $z \in X$ and z is the only fixed point of each of S and T.

PROOF : Assume that S is continuous. Then there exists a $w \in X$ such that $d(w, Sw) = \inf\{d(x, Sx) \mid x \in X\}$.
Suppose neither S nor T has a fixed point. Then

$$d(ST\,Sw, TSw) < \mathrm{Max}\left\{d(TSw, Sw), d(TSw, ST\,Sw), d(Sw, TSw), \frac{1}{2}\left[d(Sw, STSw) + d(TSw, TSw)\right]\right\}$$

so that

$$d(ST\,Sw, TSw) < d(TSw, Sw).$$

Similarly,

$$d(TSw, Sw) < d(Sw, w).$$

Thus, $d(STSw, TSw) < d(TSw, Sw) < d(Sw, w)$, a contradiction to the existence of w.
Hence either S or T has a fixed point.
Assume $Sz = z$ for some $z \in X$. Suppose $Tz \neq z$.
Then by (I),

$$d(z, Tz) = d(Sz, Tz) < d(z, Tz), \text{ a contradiction.}$$

Hence $Tz = z$. Thus $Sz = z = Tz$.
Uniqueness of common fixed point follows easily.
Since every fixed point of S is a fixed point of T and vice versa, it follows that z is the only fixed point of each of S and T.

Remark : In (I), $\frac{1}{2}[d(x, Ty) + d(y, Sx)]$ can not be replaced by Max $\{d(x, Ty), d(y, Sx)\}$ in view of the following:

Example—Let $X = \{1, 2, 3, 4\}$ with a metric d defined by $d(n, n) = 0$ for $n = 1, 2, 3, 4$, $d(1, 3) = d(1, 4) = d(2, 3) = d(2, 4) = 1$, $d(3, 4) = d(1, 2) = 2$. Let $S, T : X \to X$ be defined by $S1 = S4 = 2$, $S2 = S3 = 1$, $T1 = T3 = 4$, $T2 = T4 = 3$.

Corollary (Corollary 4 of Fisher 1980)—Let S and T be continuous mappings of a compact metric space (X, d) into itself satisfying $d(Sx, Ty) < \mathrm{Max}\ \{d(x, Sx), d(y, Ty)\}$ for all $x, y \in X$ with $Sx \neq Ty$. Then S and T have a unique common fixed point.

It is also observed that in Theorem 2 of Fisher (1980), the continuity of one of the mappings S and T is enough to ensure the existence of a unique common fixed point.

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