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# MORE ABOUT COMPACT-EXPANDABLE SPACES

D. K. THAKKAR AND DARSHAN SINGH

*Department of Mathematics, University School of Sciences, Gujarat University Ahmedabad* 380009

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(i) A space $X$ is regular and compact-expandable iff it is compact normal and compact collectionwise normal.

(ii) A regular, $T_1$ space is compact-collectionwise normal iff every discrete collection of compact subsets of $X$ can be expanded to a locally-finite collection of open subsets of $X$.

(iii) Every compact-$\chi_0$-normal space is compact-$\chi_0$ collectionwise normal.

Alas (1974, 1976, 1977) defined compact-$m$-expandable ($\sigma$-$m$-expandable), compact-$m$-normal ($\sigma$-$m$-normal) and compact-$m$-collectionwise normal spaces and obtained several consequences and characterizations of them.

In this note, Theorems 1 and 2 of Alas (1977) are improved. Moreover some characterizations of compact-collectionwise normal spaces are obtained.

All spaces are assumed to be Hausdorff. Let $m$ denote an infinite cardinal number and $\chi_0$-be the cardinality of an infinite countable set. If $A$ is a set, then the cardinality of $A$ will be denoted by $| A |$.

We say that a family $\{F_\alpha : \alpha \in A\}$ in $X$ can be expanded to a family $\{G_\alpha : \alpha \in A\}$ if $F_\alpha \subset G_\alpha$ for such $\alpha \in A$. A family $\{F_\alpha : \alpha \in A\}$ is said to be hereditarily conservative if every family $\{H_\alpha : \alpha \in A\}$ with $H_\alpha \subset F_\alpha$ for each $\alpha \in A$, is closure-preserving.

*Definition* 1 (Alas 1974)—A space $X$ is compact-$m$-expandable ($\sigma$-$m$-expandable) if for each locally-finite collection $\{F_\alpha : \alpha \in A\}$ of compact (closed, $\sigma$-compact) subsets of $X$ with $| A | \leqslant m$, there is a locally-finite collection $\{G_\alpha : \alpha \in A\}$ of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$.

In next definitions, use of '$m$' will be avoided.

*Definition* 2 (Alas 1977)—A space $X$ is compact-collectionwise normal if and only if for every discrete collection $\{F_\alpha : \alpha \in A\}$ of compact subsets of $X$, there is a mutually disjoint collection $\{G_\alpha : \alpha \in A\}$ of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$.

*Definition* 3 (Alas 1974)—A space $X$ is compact-normal if for each two disjoint closed subsets of $X$, $F$ and $K$, one of which is the union of a locally-finite family of compact subsets of $X$, there are two disjoint open sets of $X$ containing $F$ and $K$ respectively.

*Theorem* 1—A compact normal space $X$ is compact-expandable if and only if it is compact collectionwise normal.

PROOF: Sufficiency is proved in Theorem 2 of (Alas 1977). To prove necessity, let $\{F_\alpha : \alpha \in A\}$ be a discrete (hence locally-finite) collection of compact subsets of $X$. Let $\{G_\alpha : \alpha \in A\}$ be the locally-finite collection of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$.

Set $K_\alpha = G_\alpha - \bigcup_{\beta \neq \alpha} \{F_\beta\}$, $\alpha \in A$. Then $K_\alpha$ is open and $K_\alpha \cap F_\beta = \phi$ for $\beta \neq \alpha$. Also since $F_\alpha$ is compact and closed and $X$ being compact normal, there is an open set $V_\alpha$ such that

$$F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset K_\alpha, \ \alpha \in A.$$

Let $U_\alpha = V_\alpha - \bigcup_{\beta \neq \alpha} \{\bar{V}_\beta\}$. Then $\{U_\alpha : \alpha \in A\}$ is a discrete collection of open subsets of $X$ such that $F_\alpha \subset U_\alpha$, $\alpha \in A$.

In view of the above theorem and the fact that every regular, compact-expandable space is compact normal, the following theorem can be stated.

*Theorem* 2—A space $X$ is regular and compact-expandable if and only if it is compact-normal and compact-collectionwise normal.

*Theorem* 3—Let $X$ be a regular space. Then the following statements are equivalent.

(i) $X$ is compact-collectionwise normal.

(ii) If $\{F_\alpha : \alpha \in A\}$ is a locally-finite family of compact subsets of $X$ such that each point of $X$ belongs to atmost $n$ (a natural number) sets $F_\alpha$, then there is a locally-finite family $\{G_\alpha : \alpha \in A\}$ of open subsets of $X$ such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$.

(iii) Every discrete collection of compact subsets of $X$ can be expanded to a locally finite collection of open subsets of $X$.

(iv) Every discrete collection of compact subsets of $X$ can be expanded to a hereditarily conservative collection of open subsets of $X$.

PROOF : (i) $\Rightarrow$ (ii) : See the proof of Theorem 1 of (Alas 1977). (ii) $\Rightarrow$ (iii) :

If $\{F_\alpha : \alpha \in A\}$ is a discrete collection of compact subsets of $X$ then it is a locally-finite family such that each point $x$ of $X$ belongs to at most $n = 1$ sets $F_\alpha$. (iii) $\Rightarrow$ (iv) : Every locally-finite collection is hereditarily conservative.

(iv) $\Rightarrow$ (i) : Let $\{F_\alpha : \alpha \in A\}$ be a discrete collection of compact subsets of $X$ and $\{G_\alpha : \alpha \in A\}$ be the hereditarily conservative collection such that $F_\alpha \subset G_\alpha$ for each $\alpha \in A$.

Set

$$K_\alpha = G_\alpha - \bigcup_{\beta \neq \alpha} \{F_\beta\}.$$

Then $K_\alpha$ is open and $K_\alpha \cap F_\beta = \phi$ if $\alpha \neq \beta$. Since $X$ is regular, there is an open $V_\alpha$ such that

$$F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset K_\alpha.$$

Now let

$$P_\alpha = V_\alpha - \bigcup_{\beta \neq \alpha} \{\bar{V}_\beta\}.$$

Then $\{P_\alpha : \alpha \in A\}$ is the required discrete collection of open subsets of $X$ such that $F_\alpha \subset P_\alpha$ for each $\alpha \in A$.

*Remark* 4 : If $X$ is a compact normal space then all the above statements are equivalent to the fact $X$ is compact-expandable.

*Countably paracompact spaces and compact expandable spaces*

*Theorem* 5—If $X$ is countable paracompact and compact-collectionwise normal then $X$ is compact-expandable.

Proof of the above theorem is analogous to that of Theorem 2.8 (i) of Smith and Krajewski (1971).

*Corollary* 6—If $X$ is countably paracompact and compact-collectionwise normal then $X$ is $\sigma$-expandable.

PROOF : A space $X$ which is countably paracompact and compact-expandable is $\sigma$-expandable (1976).

*Theorem* 7—(i) Every countably paracompact space $X$ is $\sigma$-$\aleph_0$--expandable.

(ii) If $X$ is regular and countably paracompact then $X$ is $\sigma$-$\aleph_0$-normal.

(iii) If $X$ is compact-$\aleph_0$-normal then $X$ is compact-$\aleph_0$-collectionwise normal.

PROOF : (i) Since $X$ is countably paracompact it is compact-$\chi_0$-expandable also. Hence it is $\sigma$-$\chi_0$-expandable.

(ii) Theorem 3 (Alas 1976) states that if a space is countably paracompact and compact-$m$-normal then it is $\sigma$-$m$-normal. Also by Theorem 2 a regular countably paracompact space is compact-$\chi_0$-normal.

(iii) Let $\{F_n : n = 1, 2,\ldots\}$ be a discrete collection of closed compact subsets of $X$.

Set

$$A_1 = \bigcup_{n=2}^{\infty} \{F_n\}. \quad \text{The } F_1 \cap A_1 = \phi.$$

Since $F_1$ is compact and $A_1$ is closed there is an open set $G_1$ such that

$$F_1 \subset G_1 \text{ and } \bar{G_1} \cap A_1 = \phi.$$

Let

$$A_2 = \bar{G}_1 \cup (\bigcup_{n=3}^{\infty} \{F_n\})$$

then $F_2$ is compact and $F_2 \cap A_2 = \phi$. Hence there is an open set $G_2$ such that

$$F_2 \subset G_2 \text{ and } \bar{G_2} \cap A_2 = \phi.$$

Continuing this inductively we obtain a disjoint family $\{G_n : n = 1, 2,\ldots\}$ such that

$$F_n \subset G_n,\ n \in N.$$

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