

APPROXIMATE SOLUTIONS OF THE BOUNDARY LAYER EQUATIONS WITH SUCTION AND BLOWING

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The problem under consideration is that of boundary layer flow along a flat plate with suction (or blowing). Two types of surface suction (or blowing) are considered, namely, $v_w = \text{constant}$, and $v_w \sim x^{-1/2}$. An approximate integral method, developed by Bianchini and Co-workers (1976), is utilized. The principal merits of this method are that (i) the solutions are obtained in the closed analytical form, (ii) the similarity condition can be relaxed. The results for the velocity profiles and skin friction are presented and compared with the results of the past investigators, so that the capability of the method can be tested. In the case of suction, it is revealed that the method yields remarkably accurate results whereas for the case of blowing the method fails to locate the exact blow-off point. The method is not suitable for the cases in which the velocity profiles possess a point of inflexion—this being an inherent limitation on the method due to the assumptions involved.

1. INTRODUCTION

A simple approximate method of calculating the characteristics of the boundary layer flows has recently been described by Bianchini, Socio and Pozzi (B-S-P) (1976). The essence of this method is to assume a similarity solution even in those situations where similar solutions do not exist, and find a suitable scale factor for the similarity variable. This scale function is obtained as a solution to a single first order linear differential equation. In comparison with the highly sophisticated and time consuming computer oriented techniques, this method does not require lengthy calculations. The solutions are obtained in the closed analytical form and noteworthy, no linearization is required. B-S-P have tested this method in two extreme situations, (i) of impulsively started motion, (ii) steady flow along a flat plate. Their findings are encouraging and indicate the potential of this method for development into a useful tool for the practical calculations of boundary layer flows of more complex nature. Further exploitation and extension of this method appears warranted.

In the present paper, we are concerned with the problem of boundary layer flows along a flat plate with suction (or blowing). The classical results of this problem are discussed in standard reference books (see, for instance, Schlichting (1960), Rosenhead (1963). Further studies in boundary layer flows with suction (or blowing) using a refined Karman-Polhausen method were made by Zien (1971, 1972). He obtained results for $1/2 c_f \sqrt{Re_x}$ within 2 per cent in the entire suction region. However, his blow-off point is found to be profile dependent and found to be 0.58 and 0.67 for the two profiles used, while the actual point is 0.86.

In section 2, the B-S-P method of solving the boundary layer equations for the steady case is discussed briefly. In section 3, we consider the case of uniform suction (or blowing) while in section 4, the case of similarity suction (or blowing) is studied. A general discussion of the results appears in section 5.

2. THE GOVERNING EQUATIONS AND THE METHOD OF SOLUTION

The steady two dimensional boundary layer equations for a uniform stream could be written as

$$u_x + v_y = 0 \quad \dots(1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad \dots(2)$$

The boundary conditions are

$$u = 0, v = v_w(x) \quad \text{for } y = 0 \quad \dots(3)$$

and

$$u \rightarrow u_\infty \quad \text{as } y \rightarrow \infty$$

where $v_w(x)$ is the prescribed suction/ blowing velocity at the wall.

A brief description of the B-S-P method follows: Assume that the velocity profile has the form

$$u = u_\infty f(z) \quad \dots(4)$$

with

$$z = y/h(x). \quad \dots(5)$$

Integrating eqn. (2) across the boundary layer from zero to infinity and using the continuity equation (1), we obtain

$$u_\infty^2 \left[\int_0^\infty \frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty} \right) dy \right]_x - u_\infty v_w(x) = \gamma u_p(y=0). \quad \dots(6)$$

With the stipulation (4), the last momentum integral equation therefore becomes

$$\alpha_2 u_\infty h_x(x) - v_w(x) = \frac{v\alpha_3}{h(x)} \quad \dots(7)$$

where

$$\alpha_2 = \int_0^\infty f(1 - f) dz, \alpha_3 = f(0). \quad \dots(8)$$

A general integral of the equation gives us the scale parameter $h(x)$. The arbitrary constant in this integral is determined by adding an initial condition to the set of original boundary conditions.

A convenient choice for $f(z)$ in (4) is

$$f(z) = \operatorname{erf}(z). \quad \dots(9)$$

With this choice, from (8), we get

$$\alpha_2 = (2^{1/2} - 1)/\pi^{1/2}, \alpha_3 = 2w^{-1/2}. \quad \dots(10)$$

3. SOLUTION FOR THE CASE OF FLOW WITH UNIFORM SUCTION/BLOWING

For $v_w(x) = v_0$ (constant), eqn. (7) reduces to

$$a_2 u_\infty h_x(x) - v_0 = \frac{v\alpha_3}{h(x)}. \quad \dots(11)$$

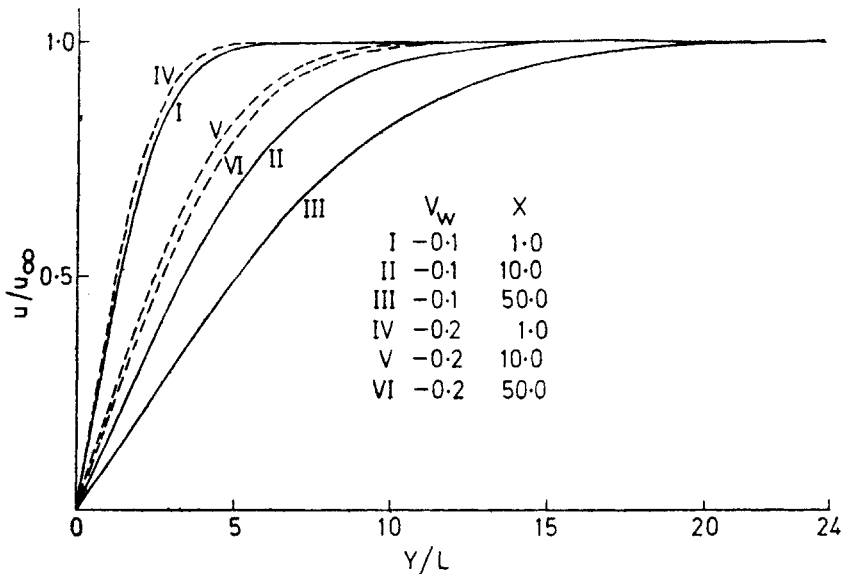


FIG. 1. Velocity profiles for uniform suction.

A solution to eqn. (11) subject to the following initial condition ... (12)

$$h(x) = 0 \text{ for } x = 0$$

is

$$h(x) + \beta \ln \left(\frac{\beta}{h(x) + \beta} \right) = \frac{\nu_0 x}{\alpha_2 u_\infty} \quad \dots (13)$$

where $\beta = \nu \alpha_3 / \nu_0$.

The skin-friction, denoted by c_f , is given by

$$\frac{1}{2} c_f \sqrt{Re_x} = \frac{\tau_w \sqrt{Re_x}}{\rho u_\infty^2} = \alpha_3 \sqrt{\frac{\nu}{u_\infty} \frac{\sqrt{x}}{h(x)}} \quad \dots (14)$$

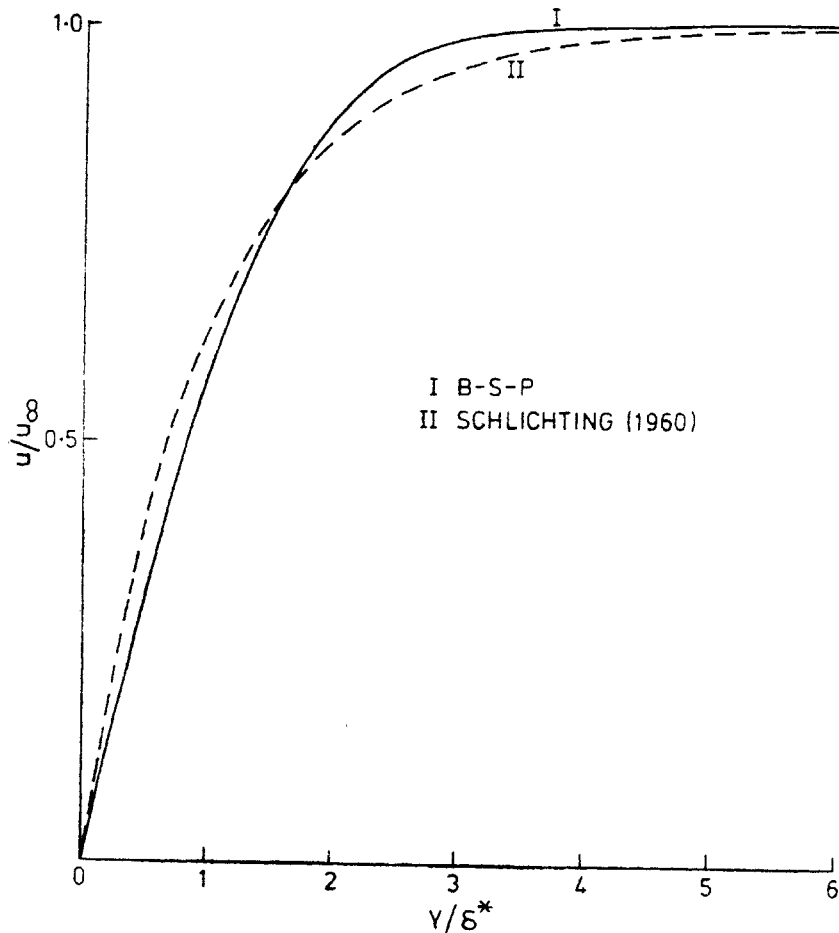


FIG. 2. Asymptotic velocity profile for uniform suction.

where $Re_x = \frac{u_\infty x}{\nu}$

and

$$\tau_w = \mu \cdot u_y (y = 0), \tag{15}$$

the shearing stress at the wall (plate).

4. SOLUTION OF THE CASE OF VARIABLE SUCTION/BLOWING

The second case we consider is

$$v_w(x) = - \frac{c}{\sqrt{x}} \tag{16}$$

where c is positive for suction and negative for blowing. Substituting $v_w(x)$ in equation (7), we obtain

$$\alpha_2 u_\infty h_x(x) + \frac{c}{\sqrt{x}} = \frac{\nu \alpha_3}{h(x)}. \tag{17}$$

A solution of the last equation subject to condition (12) is

$$h(x) = k \sqrt{x} \tag{18}$$

where

$$k = (-c \pm \sqrt{c^2 + 2\alpha_2 \alpha_3 \nu u_\infty}) / (\alpha_2 u_\infty). \tag{19}$$

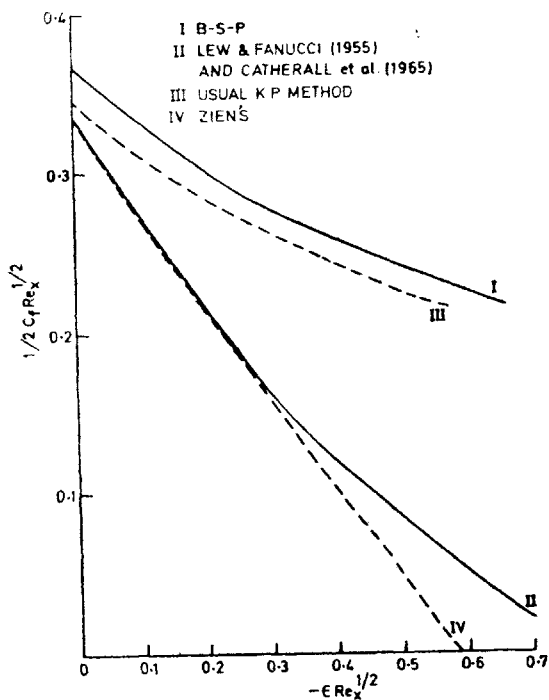


FIG. 3. Skin-friction for uniform suction.

But the negative sign before the radical in (19) has to be rejected because the scale factor has always to be positive. Thus we get

$$k = (-c + \sqrt{c^2 + 2\alpha_2 \alpha_3 u_\infty}) / (\alpha_2 u_\infty). \quad \dots(20)$$

From (18), (4) and (5), it follows that the velocity profiles for the variable suction/blowing are given by

$$u = u_\infty \operatorname{erf} \left(\frac{y}{k \sqrt{x}} \right). \quad \dots(21)$$

5. CONCLUDING REMARKS

(A) Case of Uniform Suction/Blowing

From (13), it follows that $h \rightarrow 0$ as $x \rightarrow 0$. Thus, for small x taking $h/\beta < 1$, we have

$$\ln \left(1 + \frac{h}{\beta} \right) = \frac{h}{\beta} - \frac{1}{2} \frac{h^2}{\beta^2} + \dots \quad \dots(22)$$

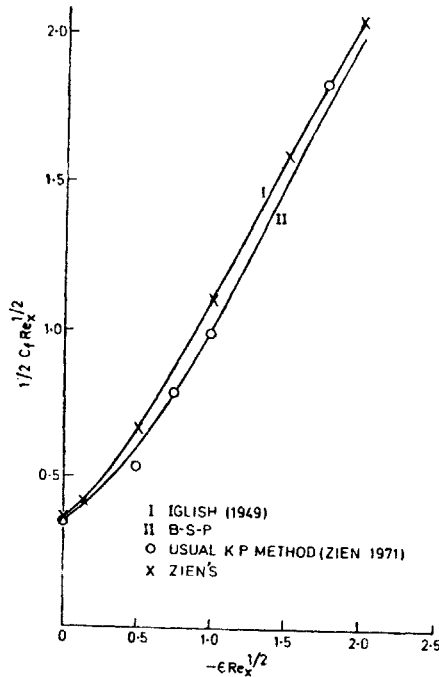


FIG. 4. Skin-friction for uniform blowing.

Substituting (22) in (13), we get

$$h - \beta \left(\frac{h}{\beta} - \frac{1}{2} \frac{h^2}{\beta^2} + \dots \right) = \frac{v_0 x}{u_\infty \alpha_2},$$

$$\therefore h - h + \frac{1}{2} \frac{h^2}{\beta} + \dots = \frac{v_0 x}{u_\infty \alpha_2}.$$

Hence, neglecting terms of order h^3/β^3 , we get

$$h^2 = \frac{2\beta v_0 x}{u_\infty \alpha_2}.$$

Thus, when x is small, the last result yields

$$h(x) = \sqrt{\frac{2v \alpha_3 x}{u_\infty \alpha_2}}. \tag{23}$$

The velocity profiles for small x are given by

$$u = u_\infty \operatorname{erf} \left(\eta / \sqrt{\frac{2v \alpha_3}{u_\infty \alpha_2}} \right) \tag{24}$$

where $\eta = y/\sqrt{x}$.

On the other hand, when x is large, in the case of uniform suction,

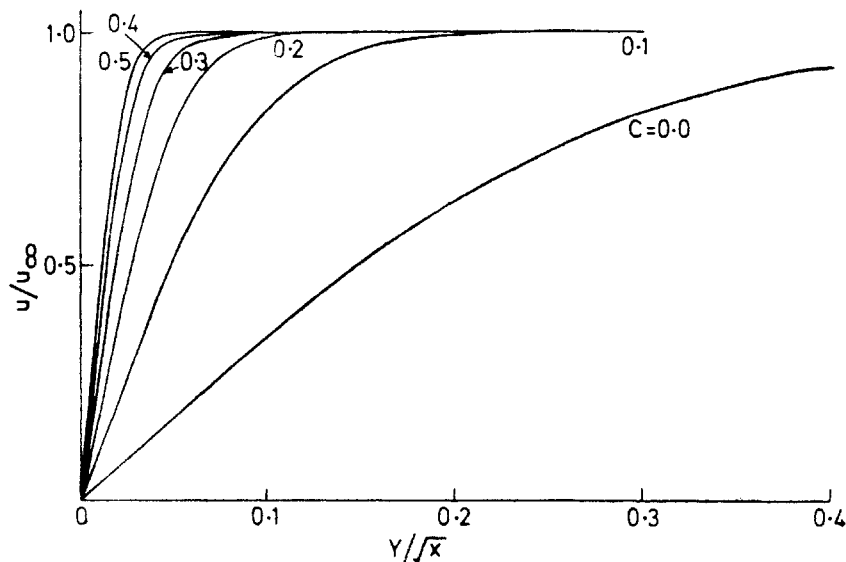


FIG. 5. Velocity profiles for similarity suction.

$$h(x) (-\beta) \left[1 - \exp \left(- \frac{v\alpha_3 x}{u_\infty \alpha_2} \right) \right] \rightarrow -\beta \text{ as } x \rightarrow \infty. \quad \dots(25)$$

The asymptotic velocity profiles for uniform suction are given by

$$\frac{u}{u_\infty} = \text{erf} (y/-\beta). \quad \dots(26)$$

Schlichting (1960) found that the asymptotic velocity profile is given by

$$\frac{u}{u_\infty} = 1 - \exp (yv_0/v). \quad \dots(27)$$

Figure 2 depicts the velocity profiles obtained from (26) and (27). It is seen that the asymptotic profile predicted by the B-S-P method is fuller as compared with the classical method. Equation (26) shows that the usual assumption of similar velocity profiles is valid for large x , but breaks down otherwise. The velocity profiles for uniform suction are presented in Fig. 1. For a fixed downstream position (i.e., for a fixed X), the velocity profiles become fuller as the suction velocity increases, as could be expected—this is observed from the curves I and IV of Fig. 1. For fixed values of v_0 , the velocity profiles become fuller as we approach the leading edge. These findings are in agreement with the work of Iglisch (1949). Opposite effects are observed in the case of blowing. The asymptotic velocity profile is attained when

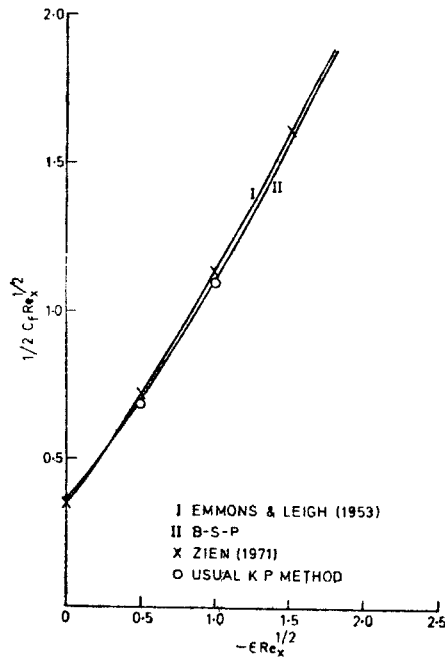


FIG. 6. Skin-friction for similarity suction.

$$\left(-\frac{v_0}{u_\infty}\right)^2 \frac{u_\infty x}{\nu} \approx 3.987$$

which compares favourably with the value obtained by Iglisch (1949), i.e.,

$$\left(-\frac{v_0}{u_\infty}\right)^2 \frac{u_\infty x}{\nu} \approx 4.$$

Next, from (14) and (15), it follows that as $x \rightarrow 0$,

$$\begin{aligned} \frac{1}{2} c_f \sqrt{Re_x} &= \sqrt{\frac{1}{2} \alpha_2 \alpha_3} \\ &= 0.3639 \end{aligned} \quad \dots(28)$$

which agrees with the value obtained by Socio and Pozzi (1979). Howarth (1938) found that $1/2 c_f \sqrt{Re_x} \approx 0.332$. The agreement is remarkably close. The last result holds for blowing as well. Figures 3 and 4 depict the variation of skin-friction with $-\epsilon \sqrt{Re_x}$ for suction and $\epsilon \sqrt{Re_x}$ for blowing, where $\epsilon = v_0/u_\infty$. It is clear that in the case of uniform suction (Fig. 3), the skin friction predicted by B-S-P method is reasonably in good agreement with that of Zien (1971) and Iglisch (1943). In general, skin friction as predicted by B-S-P method is slightly lower than that predicted by other methods mentioned earlier, while in the case of blowing, opposite

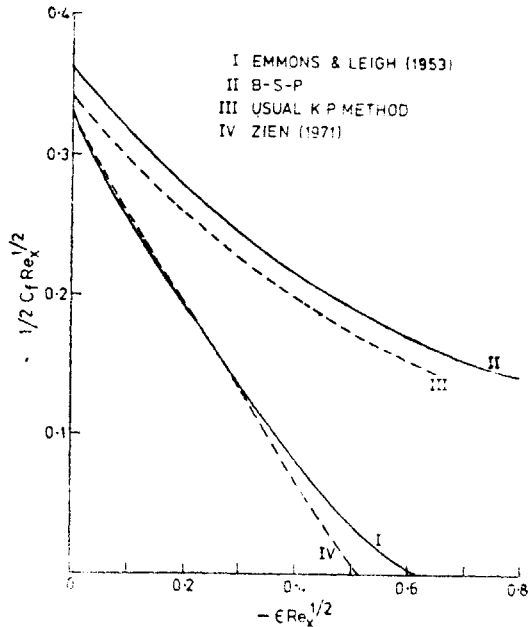


FIG. 7. Skin-friction for similarity blowing.

is true. However, B-S-P method fails to locate the separation point in the case of blowing (Fig. 4). This is due to an inherent drawback of the assumptions (4) and (9) which prohibit the velocity profiles to have a point of inflexion.

(B) Similarity Suction/Blowing

Figure 5 presents the velocity profiles for similarity suction for different values of c in eqn. (16). It is clear that as the suction velocity increases, i.e., as c increases, the velocity profiles become fuller. Figures 6 and 7 depict the variation of $1/2 c_f \sqrt{Re_x}$ with $-\epsilon \sqrt{Re_x}$ and $\epsilon \sqrt{Re_x}$ for suction and blowing respectively. Here again, we find that as in the case of uniform suction/blowing, the B-S-P method gives satisfactory results in the case of suction but fails to locate the separation point in the case of blowing.

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